Ultracontractivity and embedding into L^∞

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Abstract

Given a self-adjoint semigroup e^{-tA} satisfying an ultracontractivity bound of the type $\|e^{-tA}\|_{2\to\infty} \leq e^{m(t)}$, we find conditions on the sequence $\|A^n f\|_2^{1/n}$ that imply that f is a bounded function. Sobolev's classical embedding theorem says that, when A is the Laplace operator on $\mathbb{R}^d$, $\|A^k f\|_2 < \infty$ for some $k > d/4$ suffices to imply that f is bounded. In the cases we are interested in, the desired condition involves the whole sequence $\|A^n f\|_2^{1/n}$ and depends on the behavior of the ultracontractivity function m.

Contents

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1 Introduction

One of the classical uses of Sobolev embedding theorem is to show that an L^2 function on $\mathbb{R}^d$ having k derivatives in L^2 with $k > d/2$ is a bounded function. This has been generalized as follows. Let e^{-tA} be a semigroup of self-adjoint operators on $L^2(X,\mu)$, where (X,μ) is a σ-finite measure space. Assume that, for all $t \in (0,1)$,

$$\|e^{-tA}\|_{2\to\infty} = \sup_{\|g\|_2\le 1} \|e^{-tA}g\|_\infty \le Ct^{-\nu/4}.$$

Then any function $g \in L^2(X,\mu)$ such that $A^k g \in L^2(X,\mu)$ for some $k > \nu/4$ (roughly speaking, this corresponds to $2k$ derivatives in L^2, with $2k > \nu/2$) must be a bounded function. See, e.g., [14, Théorème 1] and the references therein.

The aim of the present paper is to obtain results in this spirit when the semigroup e^{-tA} satisfies an ultracontractivity bound of the type

$$\|e^{-tA}\|_{2\to\infty} \le e^{m(t)}, \quad t > 0 \tag{1.1}$$

with a function m which tends to infinity at least as fast as $\log 1/t$ as t tends to 0. We call such a function m an ultracontractivity function for e^{-tA}.

More precisely, we would like to obtain equivalences between (1.1) and properties such as

$$g \in \bigcap_0^\infty \mathrm{Dom}(A^n) \text{ and } \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{\phi(n)} \le 1 \Longrightarrow g \in L^\infty(X,\mu) \tag{1.2}$$

where the function m in (1.1) and the function ϕ in (1.2) are related in some explicit way. We call any function ϕ such that (1.2) holds an embedding function for A. Following the terminology of [14], we call (1.2) a generalized Gagliardo-Nirenberg inequality (although it is rather an embedding property than an inequality). Similar questions were discussed in [16] which focussed on problems related to the long time behavior of the semigroup. In this paper, the focus is on the short time behavior.

We will also relate these properties to Nash type inequalities and characterize those functions f on the real line such that $\|e^{-f(A)}\|_{2\to\infty} < \infty$. In fact, the connection between Nash inequalities and ultracontractivity bounds is sharp when the explosion of m at 0 is not too fast, whereas embedding properties of the form (1.2) give tight results when the explosion of m at 0 is fast enough. There is a common zone where both techniques give excellent results (see Theorem 5.3), and two exclusive zones where apparently only one of them stays sharp. See the examples in Section 6.

In Section 7, we consider the case where X has finite measure and A has discrete spectrum (with L^∞ bounds on the eigenfunctions), and the case of invariant operators on abelian locally compact metric groups. In these two cases, we prove sharp generalized Gagliardo-Nirenberg inequalities even when m does not belong to the favorable zone.

In the final section 8, we exhibit families of concrete examples, namely invariant diffusions on infinite dimensional tori and symmetric Lévy semigroups on the real line, which display the whole variety of behaviors considered in earlier sections.

2 Ultracontractivity functions and their transforms

This section is of a technical nature, but introduces definitions that are crucial throughout this paper. We start with a non-increasing non-negative function M, which will later coincide with the (short time) ultracontractivity function m for e^{-tA}, and define four transforms of M.

Given two non-negative functions f, g, we write $f \approx g$ to signify that there exist finite positive constants a, b such that $af \le g \le bf$ on the relevant domain. We write $f \sim g$ (say, at infinity) if $f \le (1+o(1))g$ and $g \le f(1+o(1))$.

2.1 The functions F and Φ

In this section, we set up the machinery which is necessary to connect ultracontractivity estimates with embedding properties in L^∞.

Definition 2.1 *Let M be a non-increasing non-negative function defined on $(0,+\infty)$ and such that $M(0_+) = \infty$. For non-negative x, set*

$$F(x) = F_M(x) = \inf_{t>0}\{tx + M(t)\} \tag{2.3}$$

and

$$\Phi(x) = \Phi_M(x) = \sup_{t>0}\left\{\frac{x}{t}e^{-M(t)/x}\right\}. \tag{2.4}$$

The function F is non-negative, non-decreasing, concave, equals $M(\infty)$ at 0 and tends to ∞ at ∞. It satisfies $F(x) = o(x)$ at infinity and

$$\forall\, a \in [1, \infty),\ \forall\, x \in (0, \infty), \quad F(ax) \leq aF(x). \tag{2.5}$$

The function Φ is directly related to F by

$$\Phi(x) = \sup_{y>0} \{ye^{1-\frac{F(y)}{x}}\}, \quad x > 0. \tag{2.6}$$

Indeed, we have

$$\begin{aligned} \sup_{y>0}\{ye^{1-\frac{F(y)}{x}}\} &= \sup_{y>0}\{ye^{1-\frac{\inf_{t>0}\{ty+M(t)\}}{x}}\} = \sup_{y>0}\sup_{t>0}\{ye^{1-\frac{ty}{x}-\frac{M(t)}{x}}\} \\ &= \sup_{t>0}\{\sup_{y>0}\{ye^{1-\frac{ty}{x}}\}e^{-\frac{M(t)}{x}}\} = \sup_{t>0}\left\{\frac{x}{t}e^{-\frac{M(t)}{x}}\right\} = \Phi(x). \end{aligned}$$

Lemma 2.2 *The function Φ at* (2.4) *has the following properties:*

1. *Φ is non-negative, non-decreasing, convex and satisfies $\lim_{x\to\infty}[\Phi(x)/x] = \infty$;*
2. *For $x \geq M(\infty)$, $F \circ \Phi(x) \geq x$.*

Proof of Lemma 2.2 : It is clear that Φ is non-negative and non-decreasing. It is convex because $x \mapsto xe^{-a/x}$ is convex for any $a \geq 0$. By (2.6), for any $A > 0$,

$$\Phi(x) \geq Axe^{-F(Ax)/x}.$$

Since $F(x) = o(x)$ at infinity, it follows that $\Phi(x) \geq Ax/2$ for x large enough. Hence $\Phi(x)/x$ tends to infinity. To prove the second assertion, note that $\Phi(x) \geq ye^{1-(F(y)/x)}$ for any $y > 0$ and pick y such that $F(y) = x$. Such a real y exists if $x \geq M(\infty)$.

Remarks : 1. If $M(t) \leq C_1 + C_2 \log\left(1 + \frac{1}{t}\right)$, then $\Phi(x) = \infty$ for all $x > C_2$. If $M(t) \geq c_1 + c_2 \log\left(1 + \frac{1}{t}\right)$, then $\Phi(x) < \infty$ for all $x \leq c_2$.

2. When M is convex, one can recover M from F by the Legendre inversion formula (see, e.g., [20]): $M(t) = \sup_{x\geq 0}\{-tx + F(x)\}$.

3. To see how one can retrieve M from Φ, introduce the increasing function $\widetilde{M}(t) = M(e^{-t})$, $t \in \mathbb{R}$. Setting $\widetilde{\Phi}(x) = x\log[x^{-1}\Phi(x)]$, we have $\widetilde{\Phi}(x) = \sup_{\tau\in\mathbb{R}}\{\tau x - \widetilde{M}(\tau)\}$. When $\widetilde{M}$ is convex, the Legendre inversion formula yields $\widetilde{M}(t) = \sup_{\tau\in\mathbb{R}}\{\tau t - \widetilde{\Phi}(\tau)\}$ and one can recover M from Φ by computing $\widetilde{\Phi}$, its Legendre transform $[\widetilde{\Phi}]^*$, and using the formula $M(t) = [\widetilde{\Phi}]^*(\log 1/t)$.

4. If we are given two non-negative non-increasing functions $M_1 \leq M_2$ then, with obvious notation, we have $F_1 \leq F_2$, $\Phi_1 \geq \Phi_2$. If M_1, M_2 are assumed to be convex, then $F_1 \leq F_2$ implies $M_1 \leq M_2$ by Remark 2. If $t \mapsto M_1(e^{-t}), M_2(e^{-t})$ are assumed to be convex, then $\Phi_1 \geq \Phi_2$ implies $M_1 \leq M_2$ by Remark 3.

2.2 The functions N and Q

In this section, we set up the machinery which is necessary to connect ultracontractivity estimates with Nash inequalities.

Definition 2.3 *Let M be a non-increasing non-negative function defined on $(0,+\infty)$ and such that $M(0_+)=\infty$. For any real x, set*

$$N(x)=\sup_{t>0}\{xt-tM(1/t)\}. \tag{2.7}$$

When $M\in\mathcal{C}^1$, set also

$$Q(x)=\begin{cases} -M'\circ M^{-1}(x) & \text{if } x>M(\infty)\\ 0 & \text{otherwise.}\end{cases} \tag{2.8}$$

Remarks : 1. The function N is convex and satisfies $N(x)\le 0$ for all $x\le M(\infty)$.

2. The function M is convex if and only if $t\mapsto tM(1/t)$ is convex. Thus, if M is convex, the inverse Legendre transform gives $M(t)=\sup_{x>0}\{x-tN(x)\}$.

3. When M is $\mathcal{C}^1$ and convex (hence Q monotone increasing), the function Q grows faster than x at infinity. Indeed,

$$\frac{b}{2Q(b)}\le\int_{b/2}^{b}\frac{ds}{Q(s)}=-\int_{b/2}^{b}\frac{d}{ds}[M^{-1}](s)ds=M^{-1}(b/2)-M^{-1}(b)$$

and M^{-1} tends to 0 at infinity.

4. If $M_1\le M_2$ then $N_1\ge N_2$ and, by Remark 2, the converse holds if M_1,M_2 are assumed to be convex. Note that Q does not enjoy such properties because of the use of M' in its definition.

Lemma 2.4 *Let M be a non-increasing non-negative $\mathcal{C}^1$ function defined on $(0,+\infty)$ and such that $M(0_+)=\infty$. Let F,Φ,N,Q be as in (2.3), (2.4), (2.7) and (2.8).*

1. *Fix $a,b>0$ and set $\widetilde{M}(t)=bM(at)$. The associated functions $\widetilde{F},\widetilde{\Phi},\widetilde{N},\widetilde{Q}$ are given by*

$$\widetilde{F}=aF(x/ab),\;\; \widetilde{\Phi}=ab\Phi(x/b),\;\; \widetilde{N}(x)=abN(x/b),\;\; \widetilde{Q}(x)=abQ(x/b)$$

and we have $\widetilde{F}\circ\widetilde{\Phi}(x)=aF\circ\Phi(x/b)$.

2. *Assume that M is convex. Then $Q\ge N$ and:*

   (2a) If there exists $b\in(0,\infty)$ such that

$$-tM'(t)\le bM(t)\;\; \text{for all } t \text{ small enough}, \tag{2.9}$$

   then, for any $\varepsilon>0$ we have, $Q(x)\le(b/\varepsilon)N((1+\varepsilon)x)$ for all x large enough.

(2b) If there exists $a > 0$ such that

$$-tM'(t) \geq aM(t) \text{ for all } t \text{ small enough} \tag{2.10}$$

then $F \circ \Phi(x) \leq (1 + (1/a))\, x$ for x large enough.

(2c) If both (2.9) *and* (2.10) *holds then $N \approx Q \approx \Phi$ at infinity.*

Remark : Condition (2.9) requires that M does not grow too fast at zero, whereas condition (2.10) requires that M does not grow too slowly.

Proof : Part 1 is proved by inspection. We now prove that $Q \geq N$ when M is convex. Given x, the point t_0 where the maximum of $xt - tM(1/t)$ is attained is such that $x = M(1/t_0) - (1/t_0)M'(1/t_0)$. Thus $N(x) = -M'(1/t_0)$. As M is decreasing we have $x \geq M(1/t_0)$, i.e. $M^{-1}(x) \leq 1/t_0$. Since M is convex, $-M'$ is non-increasing and $-M' \circ M^{-1}(x) \geq -M'(1/t_0) = N(x)$. Together with Remark 1, this proves that $Q \geq N$.

To prove *(2a)*, fix x large enough and $\varepsilon > 0$, and define t_1 by $x = (1+\varepsilon)M(1/t_1)$. Then $N(x) \geq \varepsilon t_1 M(1/t_1) \geq -(\varepsilon/b)M'(1/t_1)$ and thus $N(x) \geq -(\varepsilon/b)M' \circ M^{-1}(x/(1+\varepsilon))$. This gives the desired inequality.

To prove *(2b)*, for any x large enough, define $\tau = \tau(x)$ by $x = -M'(\tau)$ (note that τ goes to 0 when x goes to infinity). Then $F(x) = x\tau + M(\tau) \leq x\tau - (\tau/a)M'(\tau) = (1 + 1/a)x\tau$. To have an estimate for Φ, for x large enough, define $\sigma = \sigma(x)$ to be such that

$$(x/\sigma)e^{-M(\sigma)/x} = \sup_{s>0}(x/s)e^{-M(s)/x} = \Phi(x).$$

Such a $\sigma \in (0, \infty)$ exists because M is $\mathcal{C}^1$ and (2.10) implies that M grows faster than a positive power of $1/t$ when t tends to zero. We have $x = -\sigma M'(\sigma)$ and $\Phi(x) \leq x/\sigma = -M'(\sigma)$. Since F is non-decreasing, this gives

$$F \circ \Phi(x) \leq F(-M'(\sigma(x))) \leq (1 + (1/a))\, [-M'(\sigma(x))]\tau(-M'(\sigma(x))).$$

But $\tau(-M'(\sigma(x))) = \sigma(x)$. Hence

$$F \circ \Phi(x) \leq (1 + (1/a))\, [-M'(\sigma(x))\sigma(x)] = (1 + (1/a))\, x.$$

Finally, we prove *(2c)*. By hypothesis, we have

$$aM(t) \leq -tM'(t) \leq bM(t) \text{ for all } t \text{ small enough.} \tag{2.11}$$

As above, $\Phi(x) = (x/\sigma)e^{-(1/x)M(\sigma)}$ where $x = -\sigma M'(\sigma)$. By (2.11), this gives $\Phi(x) \approx -M'(\sigma)$ and $x \approx M(\sigma)$. As (2.11) implies that the two positive decreasing functions M, $-M'$ satisfy $M(2t) \geq cM(t)$ and $-M'(2t) \geq -cM'(t)$ for some $c > 0$ and all t small enough (i.e. M and $-M'$ are doubling near 0), we conclude that $\Phi(x) \approx -M' \circ M^{-1}(x) = Q(x)$ at infinity.

3 The eventual ultracontractivity of $e^{-sF(A)}$

Let (X, μ) be a σ-finite measure space. Recall that there is a one-to-one correspondence between semigroups $(H_t)_{t>0}$ of self-adjoint contractions on $L^2(X, \mu)$ and non-negative, possibly unbounded, self-adjoint operators A on $L^2(X, \mu)$ via the relation $H_t = e^{-tA}$, that is, $-A$ is the infinitesimal generator of H_t. Let A be a non-negative self-adjoint operator on $L^2(X, \mu)$. Let E_λ, $\lambda \in [0, \infty)$, be the spectral resolution of A so that

$$A = \int_0^\infty \lambda dE_\lambda.$$

For any function f bounded from below we can consider the bounded self-adjoint operator

$$e^{-sf(A)} = \int_0^\infty e^{-sf(\lambda)} dE_\lambda.$$

The following result relates an ultracontractivity estimate of e^{-tA} with function M to the eventual ultracontractivity of $e^{-sF(A)}$, that is, the ultracontractivity of $e^{-sF(A)}$ for s large enough.

Theorem 3.1 *Let $H_t = e^{-tA}$, $t > 0$, be a self-adjoint semigroup of contractions on $L^2(X, \mu)$. Let M and F be as in* Definition 2.1 *with M convex.*

(i) Assume that $\|H_t\|_{2\to\infty} \leq e^{M(t)}$ for all $t > 0$. Then, for any function f defined on the positive semi-axis and such that $f \geq F$, the operator $e^{-sf(A)}$ defined on $L^2(X, \mu)$ is bounded from $L^2(X, \mu)$ to $L^\infty(X, \mu)$ for all $s > 1$, uniformly on any interval (s_0, ∞), $s_0 > 1$.

(ii) Assume that for some non-negative function $f \leq F$ the operator $e^{-sf(A)}$ is bounded from $L^2(X, \mu)$ to $L^\infty(X, \mu)$ for all $s \geq s_1$ and set $C(s) = \|e^{-sf(A)}\|_{2\to\infty}$. Then we have

$$\|e^{-tA}\|_{2\to\infty} \leq C(s) e^{sM(t/s)}$$

for all $t > 0$ and all $s \geq s_1$.

For the proof, we need the following Lemma.

Lemma 3.2 *Let M and F be as in* Definition 2.1 *and set*

$$\psi(x) = \int_0^\infty e^{-xs} de^{-M(s)}.$$

For all positive x, we have

$$e^{-F(x)} \leq \psi(x) \leq (1 + F(x)) e^{-F(x)}.$$

In particular, at infinity, $-\log(\psi) \sim F$.

Proof : For fixed $x > 0$, consider the convex positive function $m_x(t) = xt + M(t)$ on $(0, \infty)$. The function m_x tends to ∞ at 0 and at ∞. Let t_x be the smallest t at which m_x attains it minimum so that $F(x) = m_x(t_x)$. We have

$$\begin{aligned} \int_0^\infty e^{-xt} de^{-M(t)} &= x \int_0^\infty e^{-(xt+M(t))} dt \geq x \int_{t_x}^\infty e^{-(xt+M(t))} dt \\ &\geq xe^{-M(t_x)} \int_{t_x}^\infty e^{-xt} dt = e^{-(xt_x+M(t_x))} = e^{-F(x)}. \end{aligned}$$

This proves the desired lower bound. For the upper bound, write

$$\begin{aligned} \int_0^\infty e^{-xt} de^{-M(t)} &= x \int_0^\infty e^{-(xt+M(t))} dt \\ &\leq x \left(\int_0^{F(x)/x} e^{-(xt+M(t))} dt + \int_{F(x)/x}^\infty e^{-xt} dt \right) \\ &\leq x \int_0^{F(x)/x} e^{-F(x)} dt + \int_{F(x)}^\infty e^{-u} du \\ &= F(x) e^{-F(x)} + e^{-F(x)}. \end{aligned}$$

That $-\log \psi \sim F$ at infinity follows easily from these two bounds.

Proof of Theorem 3.1 : We use some ideas from [16, Prop. 1.3]. To prove the first statement, set

$$\psi_t(x) = \int_0^\infty e^{-xs} de^{-tM(s/t)}.$$

Applying Lemmas 3.2 and 2.4 with $M_t(s) = tM(s/t)$, $t > 0$ fixed, shows that

$$\forall\, t > 0, \forall x > 0, \quad \psi_t(x) \geq e^{-tF(x)}.$$

As a consequence, $e^{-tf(A)} \psi_t(A)^{-1}$ is a bounded operator on $L^2(X, \mu)$ with norm less than one . Hence, for any $f \geq F$ and any $t > 1$, we have

$$\begin{aligned} \|e^{-tf(A)}\|_{2\to\infty} &= \|\psi_t(A) e^{-tf(A)} \psi_t(A)^{-1}\|_{2\to\infty} \leq \|\psi_t(A)\|_{2\to\infty} \|e^{-tf(A)} \psi_t(A)^{-1}\|_{2\to 2} \\ &\leq \int_0^\infty \|e^{-sA}\|_{2\to\infty} de^{-M_t(s)} \leq \int_0^\infty e^{M(s)} de^{-M_t(s)} \\ &\leq \int_0^\infty e^{M_t(s)/t} de^{-M_t(s)} = \int_0^{e^{-tM(\infty)}} \tau^{-1/t} d\tau = \frac{te^{-M(\infty)(t-1)}}{(t-1)}, \end{aligned}$$

which proves the claim.

To prove the second statement, let $f \leq F$ and note that

$$e^{tx} e^{-sf(x)} = e^{tx-sf(x)} \geq e^{tx-sF(x)} \geq e^{tx-s(\tau x+M(\tau))} = e^{x(t-s\tau)-sM(\tau)},$$

for all $s, t, \tau, x > 0$. In particular, for $\tau = t/s$, we get $e^{-tx}e^{sf(x)} \leq e^{sM(t/s)}$. It follows that

$$\|(e^{-sf(A)})^{-1}e^{-tA}\|_{2\to 2} \leq e^{sM(t/s)}.$$

Hence, for all $s \geq s_1$,

$$\|e^{-tA}\|_{2\to\infty} \leq \|e^{-sf(A)}\|_{2\to\infty}\|(e^{-sf(A)})^{-1}e^{-tA}\|_{2\to 2} \leq C(s)e^{sM(t/s)}.$$

This ends the proof of Theorem 3.1.

4 Embeddings into L^∞

Our first result involving the function Φ is the following theorem.

Theorem 4.1 *Let M, F, Φ be as in* Definition 2.1 *with M convex. Assume that Φ only takes finite values and that there exists $T > 0$ such that*

$$g \in \bigcap_0^\infty \mathrm{Dom}(A^n) \text{ and } \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{e^{-1}\Phi(n/T)} \leq 1 \Longrightarrow g \in L^\infty(X, \mu).$$

Then

$$\forall\, s \geq T, \quad \|e^{-sF(A)}\|_{2\to\infty} < \infty.$$

In particular, for all $t > 0$ and all $s \geq T$, there exists a constant $C(s)$ such that

$$\|e^{-tA}\|_{2\to\infty} \leq C(s)e^{sM(t/s)}.$$

Proof : Let g be in $L^2(X, \mu)$ with $\|g\|_2 = 1$. Then, spectral theory and the definition of Φ give

$$\begin{aligned} \|A^n e^{-tF(A)} g\|_2^{1/n} &\leq \left(\sup_{x>0} x^n e^{-tF(x)}\right)^{1/n} \\ &= \sup_{x>0} x e^{-(t/n)F(x)} = e^{-1}\Phi(n/t). \end{aligned}$$

Thus, the function $h = e^{-tF(A)}g$ is in $\bigcap_0^\infty \mathrm{dom}(A^n)$ and $\|A^n h\|_2^{1/n} \leq e^{-1}\Phi(n/t)$ for all n. By hypothesis, this implies that $h = e^{-tF(A)}g$ belongs to $L^\infty(X, \mu)$ if $t \geq T$. Hence, by the closed graph theorem,

$$\forall\, t \geq T, \quad C(t) = \|e^{-tF(A)}\|_{2\to\infty} < \infty.$$

By Theorem 3.1, this gives that $\|e^{-tA}\|_{2\to\infty} \leq C(s)e^{sM(t/s)}$ for all $t > 0$ and all $s \geq T$.

Our next result provides a partial converse for Theorem 4.1.

Theorem 4.2 *Let* M, F *be as in* Definition 2.1 *with* M *convex. Assume that* ϕ *is a non-negative function such that*

$$F \circ \phi(x) \leq \kappa x. \tag{4.1}$$

for some $\kappa > 0$ *and all* x *large enough. Assume further that there exists* $t_0 > 0$ *such that*

$$\|e^{-t_0 F(A)}\|_{2\to\infty} < \infty.$$

Then, for any $T > 2et_0\kappa$*, the implication*

$$g \in \bigcap_0^\infty \mathrm{Dom}(A^n) \text{ and } \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{\phi(n/T)} \leq 1 \implies g \in L^\infty(X, \mu)$$

holds true.

The proof of this theorem requires two lemmas. For any $n \geq 1$, consider the following functions.

$$M_n(t) = [M(t^{1/n})]^n, \quad t > 0; \tag{4.2}$$

$$F_n(x) = \inf_{t>0}\{xt + M_n(t)\} = F_{M_n}(x), \quad x > 0; \tag{4.3}$$

$$F_{(n)}(x) = [F(x^{1/n})]^n, \quad x > 0. \tag{4.4}$$

Lemma 4.3 *Let* M, F *be as in* Definition 2.1*. For each integer* $n \geq 1$*, the function* M_n *is non-increasing and convex. The function* F_n *is non-decreasing, concave and satisfies* $F_n(x) = o(x)$ *at infinity. Moreover*

$$F_n \leq F_{(n)} \leq 2^{n-1} F_n. \tag{4.5}$$

Proof : Computing derivative, we have

$$M_n'(t) = -t^{-(1-1/n)}[M(t^{1/n})]^{n-1}[-M'(t^{1/n})].$$

Since $t^{-(1-1/n)}$ and the two other factors are non-increasing, M_n' is non-decreasing, i.e., M_n is convex. The stated properties of $F_n = F_{M_n}$ easily follow (see Definition 2.1). The double inequality (4.5) follows easily from $a^n + b^n \leq (a+b)^n \leq 2^{n-1}(a^n + b^n)$, $a, b > 0$. The usefulness of this lemma is apparent in Lemma 4.4 below and comes from the fact that $F_{(n)}$ is not, in general, a concave function on $(0, \infty)$ (it is concave on a neighborhood of infinity, the neighborhood depending on n).

Lemma 4.4 *Let* M, F *be as in* Definition 2.1*. Let* A *be a non-negative self-adjoint operator on a Hilbert space* H*. For any* $u \in H$ *with* $\|u\| = 1$*, and any integer* n*, we have*

$$\|[F(A)]^n u\|^{1/n} \leq 2F(\|A^n u\|^{1/n}).$$

Proof : By spectral decomposition, the definition of F_{2n} and the right-hand side inequality in (4.5), we have

$$\begin{aligned}\|[F(A)]^n u\|^2 &= \int_0^\infty [F(\lambda)]^{2n} d\langle E_\lambda u, u\rangle \\ &= \int_0^\infty [F_{(2n)}(\lambda^{2n})] d\langle E_\lambda u, u\rangle \\ &\leq 2^{2n} \int_0^\infty [F_{2n}(\lambda^{2n})] d\langle E_\lambda u, u\rangle,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in H. Since F_{2n} is concave, Jensen's inequality gives

$$\begin{aligned}\|[F(A)]^n u\|^2 &\leq 2^{2n} F_{2n}\left(\int_0^\infty \lambda^{2n} d\langle E_\lambda u, u\rangle\right) \\ &= 2^{2n} F_{2n}\left(\|A^n u\|^2\right) \leq 2^{2n} F_{(2n)}\left(\|A^n u\|^2\right) \\ &= 2^{2n}\left[F\left(\|A^n u\|^{1/n}\right)\right]^{2n}.\end{aligned}$$

Here we have used (4.5) to obtain the second inequality. This proves Lemma 4.4.

Remark : Recall that a function $F : [0, \infty) \to [0, \infty)$ is a Bernstein function if F is smooth and satisfies $(-1)^{n+1} F^{(n)} \geq 0$ for $n \geq 1$ (here, $F^{(n)}$ is the n-th derivative of F). Then $F_{(n)}(x) = [F(x^{1/n})]^n$ is also a Bernstein function (see [10]). In particular, $F_{(n)}$ is concave for all n. Hence, for any Bernstein function F, the conclusion of the lemma can be improved to

$$\|[F(A)]^n u\|^{1/n} \leq F(\|A^n u\|^{1/n})$$

and, in that case, the proof follows immediately from spectral theory and Jensen's inequality. In particular, for $F(x) = x^\alpha$, $\alpha \in (0, 1)$, we have Kolmogorov's inequality

$$\|A^{\alpha n} u\| \leq \|A^n u\|^\alpha \|u\|^{1-\alpha}.$$

Proof of Theorem 4.2 : Set $T = 2et_0\kappa S$ with $S > 1$. Fix $g \in \bigcap_0^\infty \mathrm{dom}(A^n)$ such that $\|g\|_2 = 1$ (this is no loss of generality) and

$$\limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{\phi(n/T)} \leq 1.$$

Then we have

$$\|A^n g\|_2^{1/n} \leq \frac{1+S}{2}\phi(n/T) \text{ for } n \text{ large enough.} \tag{4.6}$$

To show that $g \in L^\infty(X, \mu)$, write

$$\|e^{t_0 F(A)} g\|_2 \leq \sum_0^\infty \frac{t_0^n}{n!}\|[F(A)]^n g\|_2 \leq \sum_0^\infty \frac{t_0^n}{n!}[2F(\|A^n g\|_2^{1/n})]^n$$

$$\leq \; C_1 + \sum_0^\infty \frac{t_0^n}{n!}[(1+S)F\circ\phi(n/T)]^n$$
$$\leq \; C_1 + C_2 + \sum_0^\infty \frac{((1+S)\kappa t_0 n/T)^n}{n!} = C_1 + C_2 + \sum_0^\infty \frac{(n/e)^n}{n!}\left(\frac{1+S}{2S}\right)^n < \infty.$$

Here we have used Lemma 4.4 to obtain the second inequality, (2.5) and (4.6) to obtain the third inequality and the hypothesis (4.1) to obtain the fourth inequality. The finite constants C_1, C_2 appear here to account for the finite number of terms to which both (4.6) and (4.1) do not apply because n is not large enough.

Thus $e^{t_0F(A)}g \in L^2(X,\mu)$. By hypothesis, $e^{-t_0F(A)}$ is bounded from $L^2(X,\mu)$ to $L^\infty(X,\mu)$. Hence, $g = e^{-t_0F(A)}[e^{t_0F(A)}g]$ is in $L^\infty(X,\mu)$. This ends the proof of Theorem 4.2.

We can now state one of the main theorems of this paper.

Theorem 4.5 *Let M, F, Φ be as in* Definition 2.1 *with M convex. Assume that there exists a finite positive constant κ such that*

$$F\circ\Phi(x) \leq \kappa x, \quad \textit{for all } x \textit{ large enough.} \tag{4.7}$$

Let $-A$ be the infinitesimal generator of a self-adjoint semigroup of contractions on $L^2(X,\mu)$. Then each of the properties listed below implies the one following it.

1. *$\forall\, t > 0$, $\|e^{-tA}\|_{2\to\infty} \leq e^{M(t)}$.*
2. *$\forall\, t > 1$, $\|e^{-tF(A)}\|_{2\to\infty} < \infty$.*
3. *There exists $T > 0$ such that*

$$g \in \bigcap_0^\infty \mathrm{Dom}(A^n) \textit{ and } \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{\Phi(n/T)} \leq 1 \implies g \in L^\infty(X,\mu).$$

4. *$\forall\, t \geq T$, $\|e^{-tF(A)}\|_{2\to\infty} < \infty$.*
5. *For each $s \geq T$ there exists a finite positive constant $C(s)$ such that, for all positive t, $\|e^{-tA}\|_{2\to\infty} \leq C(s)e^{sM(t/s)}$.*

This statement follows from Theorems 3.1, 4.1 and 4.2. Roughly speaking, this theorem says that properties 1, 2, 3 above are equivalent under the hypotheses of the theorem. Lemma 2.4 shows that Theorem 4.5 applies if M is convex, C^1, and satisfies (2.10).

5 Relations with Nash type inequalities

For the results in this section, we need to assume more than the mere fact that e^{-tA}, $t > 0$, is a semigroup of self-adjoint contractions. We will assume that $(e^{-tA})_{t>0}$ is sub-Markovian, i.e., e^{-tA} is self-adjoint on $L^2(X, \mu)$ and satisfies $0 \le f \le 1 \Longrightarrow 0 \le e^{-tA} f \le 1$, for all $t > 0$. In fact, what is really needed is simply that e^{-tA} acts on $L^1(X, \mu)$ and $L^\infty(X, \mu)$ with norm at most 1.

Recall the following result from [15] (the functions m, θ, $\widetilde{\theta}$ of [15] are related to our functions M, N, Q, by $\log m(t) = 2M(t)$, $\widetilde{\theta}(x) = xN(\log \sqrt{x})$, $\theta(x) = 2xQ(\log \sqrt{x})$). Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(X, \mu)$.

Proposition 5.1 *Let $-A$ be the infinitesimal generator of a sub-Markovian semigroup on $L^2(X, \mu)$. Let M, N and Q be as in* Definition 2.3.

1. [15, Pro. II.2] *Assume that, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le e^{M(t)}$. Then*
$$\forall f \in \mathrm{Dom}(A) \text{ with } \|f\|_1 \le 1, \quad \|f\|_2^2 N(\log \|f\|_2) \le \langle Af, f \rangle \tag{5.8}$$
holds true.

2. [15, Pro. II.1] *Assume that the Nash type inequality*
$$\forall f \in \mathrm{Dom}(A) \text{ with } \|f\|_1 \le 1, \quad \|f\|_2^2 Q(\log \|f\|_2) \le \langle Af, f \rangle \tag{5.9}$$
holds true. Then, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le e^{M(t)}$.

The next theorem follows from Proposition 5.1 and Lemma 2.4.

Theorem 5.2 *Let $-A$ be the infinitesimal generator of a sub-Markovian semigroup on $L^2(X, \mu)$. Let M be a C^1 convex non-increasing function defined on $(0, +\infty)$ and such that $M(0_+) = \infty$. Let N be defined by* (2.7)*. Assume that M satisfies* (2.9)*, that is, there exists b such that $-tM'(t) \le bM(t)$ for all t small enough. Then the following properties are equivalent.*

1. *There exist $c_1, c_2 \in (0, \infty)$ such that*
$$\forall t > 0, \quad \|e^{-tA}\|_{2\to\infty} \le e^{c_1 M(c_2 t)}.$$

2. *There exist $c_3, c_4 \in (0, \infty)$ such that*
$$\forall f \in \mathrm{Dom}(A) \text{ with } \|f\|_1 \le 1, \quad c_3 \|f\|_2^2 N(c_4 \log \|f\|_2) \le \langle Af, f \rangle.$$

This should be compared to [15, Theorem II.5, page 516] which is very much in the same spirit. As already mentioned above, our functions M, N, Q are precisely related to the functions $m, \widetilde{\theta}, \theta$ of [15]. However, the hypothesis made in [15] that m satisfies the condition (D) considered there is (slightly) stronger than the hypothesis that M satisfies (2.9). The conclusion of [15, Theorem II.5, page 516] is stronger than ours because it yields $c_4 = 1$.

Putting together Lemma 2.4, Theorem 4.5 and Theorem 5.2, and noting that under the assumptions below M and Φ are doubling functions, we obtain the following result.

Theorem 5.3 *Let M, F, Φ be as in* Definition 2.1. *Assume that M is C^1, convex, and that there are constants $a, b \in (0,\infty)$ such that* (2.11) *holds, that is, $aM(t) \le -tM'(t) \le bM(t)$ for all t small enough. Let $-A$ be the infinitesimal generator of a sub-Markovian semigroup on $L^2(X,\mu)$. Then the following properties are equivalent:*

1. *There exists $c_1 \in (0,\infty)$ such that, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le e^{c_1 M(t)}$.*
2. *There exists $t_0 > 0$ such that, for all $t > t_0$, $\|e^{-tF(A)}\|_{2\to\infty} < \infty$.*
3. *There exists $C_1 \in (0,\infty)$ such that for any function $f \in \bigcap_0^\infty \mathrm{Dom}(A^n)$ we have*
$$\limsup_{n\to\infty} \left\{ \frac{\|A^n f\|_2^{1/n}}{\Phi(n)} \right\} \le C_1 \Longrightarrow f \in L^\infty(X,\mu).$$
4. *There exists $C_2 \in (0,\infty)$ such that the Nash inequality*
$$\forall\, f \in \mathrm{Dom}(A) \text{ with } \|f\|_1 \le 1, \quad \|f\|_2^2 \Phi\left(\log \|f\|_2\right) \le C_2 \left(\langle Af, f\rangle + \|f\|_2^2\right)$$
is satisfied.

For another discussion related to Nash inequalities we refer to [5].

6 Discussion of specific behaviors

In this section we compute F, Φ, N, Q for some explicit functions M and spell out how the results of the previous three sections apply.

6.1 The classical case $M(t) = c + \frac{d}{4}\log(1 + \frac{1}{t})$, $c, d > 0$.

Here, the important parameter is d which plays the role of the dimension. A simple computation shows that there are constants $c_1, c_2 > 0$ depending on c, d such that
$$c_1 + \frac{d}{4}\log(1+x) \le F(x) \le c_2 + \frac{d}{4}\log(1+x).$$
Theorem 3.1(i) says that if, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le C_1(1+\frac{1}{t})^{d/4}$ then $(I + A)^{-\alpha}$ is bounded from $L^2(X,\mu)$ to $L^\infty(X,\mu)$ for all $\alpha > d/4$. Theorem 3.1(ii) says that if $(I + A)^{-d/4}$ is bounded from $L^2(X,\mu)$ to $L^\infty(X,\mu)$ then there exists a constant C_2 such that $\|e^{-tA}\|_{2\to\infty} \le C_2(1+\frac{1}{t})^{d/4}$. Compare with the somewhat sharper result in [14, Th.1].

The function Φ satisfies $\Phi \equiv \infty$ on $(d/4,\infty)$ so the results of Section 4 (except Theorem 4.2) do not apply. Theorem 4.2 gives a very weak result. Namely, by taking $\phi(x) = e^x$, it yields that if, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le C_1(1+\frac{1}{t})^{d/4}$ then any function $g \in \cap_0^\infty \mathrm{Dom}(A^n)$ satisfying
$$\limsup_{n\to\infty} e^{-n/T} \|A^n g\|_2^{1/n} \le 1$$

for some T large enough must be bounded. Theorem 3.1 gives a better result. In general, the results of Section 4 are not entirely satisfactory when applied to relatively tame ultracontractivity functions M.

The functions N and Q satisfy $N(x) \approx Q(x) \approx \exp((4/d)x)$ for x large enough. Proposition 5.1 (i.e., [15]) yields the equivalence between $\|e^{-tA}\|_{2\to\infty} \le C_1(1+\frac{1}{t})^{d/4}$ and the Nash inequality $\|f\|_2^{2(1+2/d)} \le C_1(\langle Af, f\rangle + \|f\|_2^2)\|f\|_1^{4/d}$. Note however that Theorem 5.2 only yields a weaker result. Namely, it states that a "polynomial" bound $\|e^{-tA}\|_{2\to\infty} \le C_1(1+\frac{1}{t})^{\alpha}$, for some $\alpha > 0$, is equivalent to a Nash inequality of the type $\|f\|_2^{2(1+\beta)} \le C_1(\langle Af, f\rangle + \|f\|_2^2)\|f\|_1^{2\beta}$ for some $\beta > 0$. The correspondence between α and β has been lost in this statement (one should have $\beta = 2/\alpha$).

6.2 The case $M(t) = c_1 + c_2[\log(1+\frac{1}{t})]^{\lambda}$, $c_1, c_2 > 0$, $\lambda > 1$.

Somewhat tedious computations show that

$$F(x) \approx 1 + [\log(1+x)]^{\lambda}, \quad \Phi(x) \approx x \exp\left(\left(1 - \frac{1}{\lambda}\right)\left(\frac{x}{c_2\lambda}\right)^{1/(\lambda-1)}\right).$$

Theorem 3.1 yields the equivalence between the ultracontractivity property

$$\exists\, c_1, c_2 > 0,\ \forall\, t > 0, \quad \|e^{-tA}\|_{2\to\infty} \le e^{c_1 + c_2[\log(1+\frac{1}{t})]^{\lambda}}, \tag{6.10}$$

and

$$\exists\, t_0 > 0,\ \forall\, t > t_0,\ \ \|e^{-t[\log(I+A)]^{\lambda}}\|_{2\to\infty} < \infty.$$

Theorem 4.5 does not apply because, for large x, $F \circ \Phi(x) \approx x^{\lambda/(\lambda-1)}$ and thus condition (4.7) is not satisfied. Nevertheless, we can apply Theorem 4.1 and Theorem 4.2. Theorem 4.1 says that, if there exists T such that

$$g \in \bigcap_0^{\infty} \mathrm{Dom}(A^n) \text{ and } \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{e^{(n/T)^{1/(\lambda-1)}}} \le 1 \implies g \in L^{\infty}(X, \mu),$$

then there are constants $c_1, c_2 > 0$ such that (6.10) holds. Theorem 4.2 says that, if there are constants $c_1, c_2 > 0$ such that (6.10) holds, then there exists T such that

$$g \in \bigcap_0^{\infty} \mathrm{Dom}(A^n) \text{ and} \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{e^{(n/T)^{1/\lambda}}} \le 1 \implies g \in L^{\infty}(X, \mu).$$

Thus we have not quite obtained the desired equivalence between ultracontractivity and embedding in this case. Again, this reflects the unsatisfactory nature of the results of Section 4 when applied to tame functions M.

The functions N, Q satisfy

$$N(x) \approx Q(x) \approx x^{(\lambda-1)/\lambda} \exp\left((x/c_2)^{1/\lambda}\right).$$

Theorem 5.2 gives the equivalence between the ultracontractivity property (6.10) and the Nash type inequality

$$\forall f \in \mathrm{Dom}(A) \text{ with } \|f\|_1 \leq 1, \;\; c_3\|f\|_2^2 \exp(c_4[\log(1+\|f\|_2)]^{1/\lambda}) \leq \langle Af, f\rangle + \|f\|_2^2.$$

In this statement, no information is given on the relation between the important constants c_2, c_4. The computation of N and Q above and a direct application of Proposition 5.1 yield a more precise result.

6.3 The case $M(t) = c_1 + t^{-\lambda}$, $c_1, \lambda > 0$.

This is, in a sense, typical of the cases we want to consider in this work. Section 6.5 treats the more general case when M is regularly varying. Set $\lambda' = \lambda/(1+\lambda)$. One easily computes

$$F(x) = c_1 + (1+\lambda)x^{\lambda'}, \;\; \Phi(x) = \lambda^{-1/\lambda} e^{-\lambda(1+c_1/x)} x^{1/\lambda'}$$

and

$$N(x) = \lambda\left(\frac{x-c_1}{1+\lambda}\right)^{1/\lambda'}, \;\; Q(x) = \lambda\,(x-c_1)^{1/\lambda'}, \;\; x \geq c_1.$$

Theorems 3.1, 4.1, 4.2, 4.5, 5.2 all apply as well as Theorem 5.3 which, for sub-Markovian semigroups, states the equivalence of the following properties:

1. There exists $c_1 > 0$ such that, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \leq e^{c_1 + t^{-\lambda}}$;
2. There exists $t_0 \in (0,\infty)$ such that, for all $t > t_0$, $\|e^{-tA^{\lambda'}}\|_{2\to\infty} < \infty$;
3. There exists a finite positive constant C such that for any $f \in \bigcap_0^\infty \mathrm{Dom}(A^n)$,

$$\limsup_{n\to\infty}\left\{n^{-1/\lambda'}\|A^n f\|_2^{1/n}\right\} \leq C \implies f \in L^\infty(X,\mu).$$

4. There exists $C_1 \in (0,\infty)$ such that for all $f \in \mathrm{Dom}(A)$ with $\|f\|_1 \leq 1$, we have

$$\|f\|_2^2[\log(1+\|f\|_2)]^{1/\lambda'} \leq C_1(\langle Af, f\rangle + \|f\|_2^2).$$

6.4 The case $M(t) = c_1 e^{c_2[\log(1+\frac{1}{t})]^\lambda}$, $c_1, c_2, \lambda > 0$.

This is really two different cases depending on whether $\lambda \in (0,1)$ or $\lambda \in (1,\infty)$. For $\lambda = 1$, it reduces to $M(t) = (1+\frac{1}{t})^{c_2}$ which is essentially the case treated in Section 6.3. For any fixed $\lambda \in (0,1)$, $\log F(x) \sim c_2[\log(1+x)]^\lambda$ and there are constants $c_3, c_4 > 0$ such that

$$c_3 \exp\left(c_3(\log(1+x)]^\lambda\right) \le F(x) \le c_4 \exp\left(c_4(\log(1+x)]^\lambda\right).$$

When instead $\lambda \in (1,\infty)$, there exist constants $c_3, c_4 > 0$ such that

$$c_3\,(1+x)\,\exp\left(-c_4(\log(1+x)]^{1/\lambda}\right) \le F(x) \le c_4\,(1+x)\,\exp\left(-c_3(\log(1+x)]^{1/\lambda}\right).$$

Theorem 3.1 applies but we will not write the result explicitely.

Concerning the function Φ, there are $c_5, c_6 > 0$ such that

$$x\exp\left(c_5(\log x)^{1/\lambda}\right) \le \Phi(x) \le x\exp\left(c_6(\log x)^{1/\lambda}\right)$$

for x large enough. When $\lambda \in (0,1)$, tedious computations show that $x^{-1}F \circ \Phi(x)$ tends to infinity at infinity, hence Theorem 4.5 does not apply. When $\lambda \ge 1$, (2.10) holds true, $F \circ \Phi(x) \approx x$, and Theorem 4.5 applies. For any $\lambda \in (0,\infty)$, we can still apply Theorems 4.2 and 4.1 which actually give a quite satisfactory result stated below in Theorem 6.1.

Computing N and Q, one finds that, for large x,

$$Q(x) \approx x\,(\log x)^{(\lambda-1)/\lambda}\exp\left(\left(c_2^{-1}\log(x/c_1)\right)^{1/\lambda}\right)$$

and that there are constants $c_7, c_8 \in (0,\infty)$ such that

$$x\exp\left(c_7\,(\log x)^{1/\lambda}\right) \le N(x) \le x\exp\left(c_8\,(\log x)^{1/\lambda}\right).$$

If $\lambda \in (0,1)$, Lemma 2.4 and Theorem 5.2 apply. If $\lambda > 1$, one can check that N is substantially smaller than Q in the sense that, for any fixed $A > 1$, $N(Ax) = o(Q(x))$ at infinity.

As (2.11) fails if $\lambda \ne 1$, Theorem 5.3 does not apply. However, Theorems 4.1, 4.2 and Proposition 5.1 yield the following result (recall that $\log_{(2)}(s) = \log(1+\log(1+s))$).

Theorem 6.1 *Fix $\lambda \in (0,\infty)$. Let $-A$ be the infinitesimal generator of a sub-Markovian semigroup on $L^2(X,\mu)$. The following properties are equivalent:*

1. *There exist $c_1, c_2 \in (0,\infty)$ such that, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le e^{c_1\exp(c_2[\log(1+\frac{1}{t})]^\lambda)}$.*
2. *There exists $c_3 \in (0,\infty)$ such that for any function $f \in \bigcap_0^\infty \mathrm{Dom}(A^n)$ we have*
$$\limsup_{n\to\infty}\left\{\frac{\|A^n f\|_2^{1/n}}{n e^{c_3(\log n)^{1/\lambda}}}\right\} < \infty \implies f \in L^\infty(X,\mu).$$
3. *There exist $c_4, c_5 \in (0,\infty)$ such that for all $f \in \mathrm{Dom}(A)$ with $\|f\|_1 \le 1$, we have*
$$c_4\|f\|_2^2\log(1+\|f\|_2)\exp\left(c_5\left(\log_{(2)}\|f\|_2\right)^{1/\lambda}\right) \le \langle Af, f\rangle + \|f\|_2^2.$$

Note that this statement does not give information on the relationships between the important constants c_2, c_3, c_5. This reflects in part the fact that Theorem 5.3 does not apply.

6.5 Regular variation

This section generalizes the example $M(t) = c_1 + t^{-\lambda}$ by treating functions of regular variation. Recall the following classic definition (see [13]). A measurable positive function f is of regular variation of index λ (at infinity), $\lambda \in \mathbb{R}$, if, for all $\rho > 0$,

$$\lim_{x\to\infty} f(\rho x)/f(x) = \rho^{\lambda}.$$

The class of these functions is denoted by $\mathbf{R}_\lambda$. Functions in $\mathbf{R}_0$ are called slowly varying functions.

A positive function f is of smooth regular variation of order λ (i.e., $f \in \mathbf{SR}_\lambda$) if $h(x) = \log f(e^x)$ is smooth in a neighborhood of infinity and

$$\lim_{x\to\infty} h'(x) = \lambda, \quad \lim_{x\to\infty} h^{(n)}(x) = 0 \text{ for all } n = 2, 3, \ldots.$$

According to [13, (1.8.1')], a function f is in $\mathbf{SR}_\lambda$ if and only if it is smooth in a neighborhood of infinity and

$$\lim_{x\to\infty} x^n f^{(n)}(x)/f(x) = \lambda(\lambda-1)...(\lambda-n+1) \text{ for all } n = 1, 2, \ldots.$$

By [13, Th 1.8.2], for any $f \in \mathbf{R}_\rho$ there exist $f_1, f_2 \in \mathbf{SR}_\rho$ such that $f_1 \sim f_2$ and $f_1 \le f \le f_2$ in some neighborhood of infinity. For $\lambda, \alpha, \beta \in \mathbb{R}$, the function $f(t) = t^\lambda (\log t)^\alpha (\log\log t)^\beta$ is an example of a function in $\mathbf{R}_\lambda$.

Proposition 6.2 *Let M, F, Φ be as in* Definition 2.1. *Assume that $t \mapsto M(1/t) \in \mathbf{SR}_\lambda$ for some $\lambda > 0$. Set $\lambda' = \lambda/(1+\lambda)$ and $M_1(t) = t^{-1}M(t)$. Then*

1. *$F \in \mathbf{SR}_{\lambda'}$, in fact there exists $a_\lambda > 0$ such that $F \sim a_\lambda M \circ M_1^{-1}$.*
2. *$\Phi \in \mathbf{SR}_{1/\lambda'}$, in fact there exists $b_\lambda > 0$ such that $\Phi(x) \sim b_\lambda x/M^{-1}(x)$.*
3. *There exists $c_\lambda > 0$ such that $F \circ \Phi(x) \sim c_\lambda x$ at infinity.*
4. *$N \approx Q \approx \Phi$ in a neighborhood of infinity. In fact, there exist $d_\lambda, d'_\lambda > 0$ such that $N \sim d_\lambda \Phi$, $Q \sim d'_\lambda \Phi$.*

Proof : The function $t \mapsto xt + M(t)$ attains its infimum at τ satisfying $x = -M'(\tau)$. As $tM'(t) \sim -\lambda M(t)$ as t tends to 0, we have $\tau x \sim \lambda M(\tau)$ (as x tends to infinity). This implies

$$F(x) = \tau x + M(\tau) \sim (1+\lambda)M(\tau) \sim (1+\lambda)M \circ [-M']^{-1}(x).$$

It is well known that $M \in \mathbf{SR}_{-\lambda}$ implies $-M' \in \mathbf{SR}_{-1-\lambda}$ and $[-M']^{-1} \in \mathbf{SR}_{-1/(1+\lambda)}$. Hence $F \in \mathbf{SR}_{\lambda/(1+\lambda)}$. This proves the first statement. To prove the second and third statements, consider the function $y \mapsto ye^{1-(1/x)F(y)}$. It attains its supremum at at point z such that $x = zF'(z)$. As $zF'(z) \sim \lambda' F(z)$, we have $F(z) \sim x/\lambda'$ and

$$\Phi(x) \sim e^{1-(1/\lambda')} z \sim e^{1-(1/\lambda')} F^{-1}(x/\lambda') \sim e^{1-(1/\lambda')} (1/\lambda')^{1/\lambda'} F^{-1}(x).$$

Hence, $\Phi(x) \in \mathbf{SR}_{1/\lambda'}$ and $F \circ \Phi(x) \sim c_\lambda x$. The last assertion follows from the proof of Lemma 2.4(2c).

Theorem 5.3 applies in this setting and yields the following result.

Theorem 6.3 *Let M, F, Φ be as in* Definition 2.1 *with M convex. Assume further that $t \mapsto M(1/t) \in \mathbf{SR}_\lambda$ for some $\lambda > 0$. Set $M_1(t) = t^{-1}M(t)$. Let $-A$ be the infinitesimal generator of a sub-Markovian semigroup on $L^2(X, \mu)$. Then the following properties are equivalent:*

1. *There exists $c_1 \in (0, \infty)$ such that, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le e^{c_1 M(t)}$.*
2. *There exists $t_0 > 0$ such that, for all $t > t_0$, $\|e^{-tM \circ M_1^{-1}(I+A)}\|_{2\to\infty} < \infty$.*
3. *There exists $C_1 \in (0, \infty)$ such that for any function $f \in \bigcap_0^\infty \mathrm{Dom}(A^n)$ we have*

$$\limsup_{n\to\infty} \left\{ \frac{\|A^n f\|_2^{1/n}}{n/M^{-1}(n)} \right\} \le C_1 \implies f \in L^\infty(X, \mu).$$

4. *There exists $C_2 \in (0, \infty)$ such that the Nash inequality*

$$\forall\, f \in \mathrm{Dom}(A) \text{ with } \|f\|_1 \le 1, \quad \|f\|_2^2 \frac{\log \|f\|_2}{M^{-1}(\log \|f\|_2)} \le C_2 \left(\langle Af, f \rangle + \|f\|_2^2 \right)$$

   is satisfied.

In order to apply this theorem in concrete cases, one needs to compute M^{-1} and M_1^{-1}. This can be done by using Proposition 1.5.15 and Proposition 2.3.5 in [13]. The following examples illustrate such computations. Set $\log_{(1)}(t) = \log(1+t)$ and

$$\log_{(k)}(t) = \log(1 + \log_{(k-1)}(t)). \tag{6.11}$$

Proposition 6.4 *Fix $\lambda > 0$ and set $\lambda' = \lambda/(1+\lambda)$. Assume that $M(t) = t^{-\lambda}\ell(1/t)$ where ℓ is a slowly varying function.*

1. *Assume that $\ell(t) = \prod_1^k (\log_{(i)} t)^{\beta_i}$. Then*

$$F(x) \sim x^{\lambda'} \ell(x)^{\lambda'/\lambda}, \;\; \Phi(x) \sim x^{1/\lambda'} \ell(x)^{-1/\lambda} \text{ at infinity.}$$

2. *Assume that* $\ell(t) = \exp\left(\varepsilon(\log t)^{\beta}\right)$ *with* $\beta \in (0, 1/2)$ *and* $\varepsilon = \pm 1$. *Then*

$$F(x) \sim x^{\lambda'} \exp\left(\varepsilon(1+\lambda)^{-1-\beta}(\log x)^{\beta}\right)$$

and

$$\Phi(x) \sim x^{1/\lambda'} \exp\left(-\varepsilon\lambda^{-1-\beta}(\log x)^{\beta}\right) \text{ at infinity.}$$

3. *Assume that* $\ell(t) = \exp\left(\varepsilon(\log t)^{\beta}\right)$ *with* $\beta \in (1/2, 1)$ *and* $\varepsilon = \pm 1$. *Then there are positive finite constants* c_i, $1 \leq i \leq 4$, *depending on* λ *and* β, *such that, at infinity,*

$$x^{\lambda'} \exp\left(\varepsilon c_1(\log x)^{\beta}\right) \leq F(x) \leq x^{\lambda'} \exp\left(\varepsilon c_2(\log x)^{\beta}\right)$$

and

$$x^{1/\lambda'} \exp\left(-\varepsilon c_3(\log x)^{\beta}\right) \leq \Phi(x) \leq x^{1/\lambda'} \exp\left(-\varepsilon c_4(\log x)^{\beta}\right).$$

Proof : These results involve tedious computations. A basic tool is the asymptotic inverse formula of [13, Prop. 1.5.15]. For the first statement, one also uses [13, Prop. 2.3.5] (see [13, Ex. 1, p. 433]). For the second statement, one uses [13, Ex. 3, p. 435]. The third statement, which is less precise, is obtained by inspection.

6.6 The case $M(t) = \exp(g(1/t))$, $g \in \mathbf{SR}_{\lambda}$, $\lambda > 0$.

This is a case where M is a rapidly varying function. The proof of the following proposition is along the same lines as the proof of Proposition 6.2.

Proposition 6.5 *Let* M, F *be as in* Definition 2.1. *Let* Φ *be as in* (2.6). *Assume that* $M(t) = \exp(g(1/t))$ *where* $g \in \mathbf{SR}_{\lambda}$ *for some* $\lambda > 0$. *Then*

1. $F(x) \sim x/g^{-1}(\log x)$ *at infinity;*
2. $\Phi(x) \sim xg^{-1}(\log x)$ *at infinity;*
3. $F \circ \Phi(x) \sim x$ *at infinity.*

Proposition 1.5.15 and Proposition 2.3.5 in [13] show how to compute g^{-1}. Proposition 6.4 above gives formulas in some special cases. Theorem 4.5 applies nicely in this setting and yields the following.

Theorem 6.6 *Assume that* $g \in \mathbf{SR}_{\lambda}$ *for some* $\lambda > 0$. *Let* $-A$ *be the infinitesimal generator of a self-adjoint semigroup of contractions on* $L^2(X, \mu)$. *Then the following properties are equivalent.*

1. *There exists* $c > 0$ *such that* $\forall\, t > 0$, $\|e^{-tA}\|_{2\to\infty} \leq e^{\exp(cg(1/t))}$.
2. *There exists* $t_0 > 0$ *such that* $\|e^{-t_0 \frac{(I+A)}{g^{-1}(\log(I+A))}}\|_{2\to\infty} < \infty$.

3. There exists a constant $C > 0$ such that

$$u \in \bigcap_0^\infty \mathrm{Dom}(A^n) \text{ and } \limsup_{n\to\infty} \frac{\|A^n u\|_2^{1/n}}{ng^{-1}(\log n)} \le C \implies u \in L^\infty(X,\mu).$$

Note that this is a case when Nash techniques break down because N/Q tends to 0 at infinity (see [17, Example 2.3.5]). Indeed, $N(x) \approx xg^{-1}(\log x)$ whereas $Q(x) \approx x(\log x)g^{-1}(\log x)$.

7 Further results concerning tame behaviors

This section presents a different way of relating the behavior of "ultracontractivity functions" to that of "embedding functions". If M is as in Definition 2.1 and Φ is defined by (2.4), Theorem 4.1 states that the embedding property (1.2) with $\phi(x) = \Phi(x)$ implies the ultracontractivity bound (1.1) with $m(t) = M(t) + c$ for some constant c. Theorem 4.5 provides a converse only in the case when (4.7) holds, that is, when there exists $\kappa > 0$ such that

$$F \circ \Phi(x) \le \kappa x.$$

This condition excludes the case where the behavior of M is "tame". For instance, (4.7) does not hold when $M(t) = c_1 + c_2[\log(1 + 1/t)]^\lambda$, $c_1, c_2 > 0$, $\lambda > 1$. See Section 6.2.

Under additional hypotheses on the infinitesimal generator A, this section gives an equivalence between (1.1) and (1.2) that covers more or less exactly the cases where Theorem 4.5 does not apply.

7.1 Probability spaces with polynomially bounded eigenfunctions

Let e^{-tA}, $t > 0$, be a semigroup of self-adjoint contractions on $L^2(X,\mu)$. In this section, we assume that (X,μ) is a probability space and that there exists a countable orthonormal basis $(u_i)_0^\infty$ of $L^2(X,\mu)$ such that $u_i \in \mathrm{Dom}(A)$ and $Au_i = \lambda_i u_i$ with $0 \le \lambda_0 \le \lambda_1 \le \ldots < \infty$ and $\lim_{i\to\infty} \lambda_i = \infty$ (eigenvalues are repeated according to their multiplicity). We set

$$r(i) = \#\{j : \lambda_j \le \lambda_i\}. \tag{7.12}$$

Thus, if all the eigenvalues λ_i are distincts, we have $r(i) = 1 + i$ but if the multiplicity of the eigenvalues grows with i, $r(i)$ can be much larger than i. We consider the condition

$$\text{there exist } C, a \ge 0 \text{ such that } \|u_i\|_\infty \le Cr(i)^a, \tag{7.13}$$

which may or may not be satisfied.

Theorem 7.1 *Let (X,μ) and $(e^{-tA})_{t>0}$ be as described above. Let M, Φ be as in* Definition 2.1*. Assume that M is $\mathcal{C}^1$, that $t \mapsto -tM'(t)$ is decreasing on $(0,\infty)$ and tends to ∞ when t*

tends to 0*. Assume that there exists* $b > 0$ *such that* (2.9) *holds, that is,* $m(t) \leq bM(t)$ *for all* t *small enough, and that* Φ *does not take the value* $+\infty$*. Assume also that condition* (7.13) *is satisfied. If, for all* $t > 0$*, we have* $\|e^{-tA}\|_{2\to\infty} \leq e^{M(t)}$ *then, for any* $T > 2(1+a)(1+b)$*,*

$$g \in \bigcap_0^\infty \mathrm{Dom}(A^n) \text{ and } \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{\Phi(n/T)} \leq 1 \implies g \in L^\infty(X,\mu).$$

We will need the following lemmas.

Lemma 7.2 *Let* (X,μ) *and* $(e^{-tA})_{t>0}$ *be as described above. Let* M *be as in* Definition 2.1 *and* N *be the Nash function associated to* M *by* (2.7)*. Assume that* $\|e^{-tA}\|_{2\to\infty} \leq e^{M(t)}$*. Then, for any integer* k*,*

$$\lambda_k \geq N\left(\log\sqrt{r(k)}\right).$$

Proof : By the spectral and ultracontractivity hypotheses concerning A, the semigroup e^{-tA} admits a bounded kernel $h_t(x,y)$ which is given by

$$h_t(x,y) = \sum_0^\infty e^{-t\lambda_i} u_i(x)\overline{u_i}(y)$$

and satisfies $h_{2t}(x,x) = \|h_t(x,\cdot)\|_2^2 \leq \|e^{-tA}\|_{2\to\infty}^2 \leq e^{2M(t)}$. Using the hypothesis that $\mu(X) = 1$, we obtain

$$\int_M h_{2t}(x,x)d\mu(x) = \sum_0^\infty e^{-2t\lambda_i} \leq e^{2M(t)}.$$

By Definition (7.12), this implies

$$r(k)e^{-2t\lambda_k} \leq e^{2M(t)}.$$

Thus, for any $t > 0$, $\lambda_k \geq t^{-1}\left(\log\sqrt{r(k)} - M(t)\right)$ or, equivalently

$$\lambda_k \geq \sup_{\tau>0}\left\{\tau\log\sqrt{r(k)} - \tau M(1/\tau)\right\} = N\left(\log\sqrt{r(k)}\right).$$

Lemma 7.3 *Assume that* M *and* Φ *are as in* Theorem 7.1 *and* N *as in* (2.7)*. Then, for any fixed* $K, y_0, t > 0$*,*

$$\limsup_{x\to\infty} \frac{1}{x} \min_{y>y_0}\left\{\frac{y^2}{y+t}\log\left(\frac{K\Phi(y)}{N(x)}\right)\right\} \leq -(1+b)^{-1}K^{-b}.$$

Proof : From the definition of N in terms of M, it follows that, for each x large enough, there exists $t_0 = t_0(x) \in (0, \infty)$ at which the maximum defining $N(x)$ is attained; this t_0 satisfies

$$x = -t_0 M'(t_0) + M(t_0), \quad N(x) = -M'(t_0) \tag{7.14}$$

and tends to 0 when x tends to infinity. In particular, for x large enough, (2.9) implies $-t_0 M'(t_0) < bM(t_0)$.

For Φ, if y is such that $\Phi(y)$ is finite, then there exists $t_1 = t_1(y) \in (0, \infty)$ at which the maximum defining Φ is attained and this t_1 satisfies

$$y = -t_1 M'(t_1), \quad \Phi(y) = -M'(t_1) e^{\frac{M(t_1)}{t_1 M'(t_1)}}. \tag{7.15}$$

Because we assume that $t \to -tM'(t)$ is decreasing, $t_1(y)$ is uniquely determined by the equation $y = -t_1 M'(t_1)$.

Now, fix x large enough. Let t_0 be as above and set $y = -t_1 M'(t_1)$ with $t_1 = Kt_0$. Then

$$\Phi(y) = -M'(t_1) e^{\frac{M(t_1)}{t_1 M'(t_1)}}.$$

As m is decreasing, we have $-KM'(t_1) \le -M'(t_0) = N(x)$. Thus

$$y \log \left(\frac{K\Phi(y)}{N(x)} \right) \le -M(t_1).$$

By (2.9), for small enough t_0 (i.e., large enough x), we have $M(t_0) \le K^b M(Kt_0)$ and thus $x = -t_0 M'(t_0) + M(t_0) \le (1+b)M(t_0) \le (1+b)K^b M(t_1)$. It follows that

$$\frac{y^2}{y+t} \log \frac{K\Phi(y)}{N(x)} \le -\frac{y}{y+t}(1+b)^{-1} K^{-b} x.$$

Since y tends to infinity with x, we conclude that

$$\limsup_{x\to\infty} \frac{1}{x} \min_{y>y_0} \left\{ \frac{y^2}{y+t} \log \left(\frac{\Phi(y)}{N(x)} \right) \right\} \le -(1+b)^{-1} K^{-b}$$

as desired.

Proof of Theorem 7.1 : We want to show that for any $T > 2(1+a)(1+b)$, the condition

$$g \in \bigcap_0^\infty \mathrm{Dom}(A^n) \text{ and } \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{\Phi(n/T)} \le 1 \tag{7.16}$$

implies that $\|g\|_\infty < \infty$. For any function $g \in L^2(X, \mu)$, let $g = \sum g_k u_k$, $g_k = \langle g, u_k \rangle$, be the expansion of g along the orthonormal basis $(u_i)_0^\infty$. We have $\|A^n g\|_2^2 = \sum_0^\infty |g_k|^2 \lambda_k^{2n}$ and thus, if g satisfies (7.16), for any $K > 1$, there exists n_K such that for all $n > n_K$ we have

$$|g_k| \le e^{n \log\left(\|A^n g\|_2^{1/n} / \lambda_k \right)} \le e^{n \log\left(K\Phi(n/T)/N\left(\log \sqrt{r(k)}\right)\right)} \tag{7.17}$$

where we have used Lemma 7.2 and the assumption on g to obtain the second inequality. Now, let n, z be such that $n \le z < n+1$ and $K\Phi(n/T)/N(\log\sqrt{r(k)}) \le 1$. Then, we have

$$n \log\left(\frac{K\Phi(n/T)}{N\left(\log\sqrt{r(k)}\right)}\right) \le \frac{z^2}{z+1}\log\left(\frac{K\Phi(z/T)}{N\left(\log\sqrt{r(k)}\right)}\right).$$

Hence, for all k large enough,

$$\begin{aligned}\inf_{n>n_K}\left\{n\log\left(\frac{K\Phi(n/T)}{N\left(\log\sqrt{r(k)}\right)}\right)\right\} &\le \inf_{y>n_K}\left\{\frac{y^2}{y+1}\log\left(\frac{K\Phi(y/T)}{N\left(\log\sqrt{r(k)}\right)}\right)\right\}\\ &= T\inf_{y>n_K/T}\left\{\frac{y^2}{y+T}\log\left(\frac{K\Phi(y)}{N\left(\log\sqrt{r(k)}\right)}\right)\right\}.\end{aligned}$$

By Lemma 7.3, given $\varepsilon \in (0,1)$ there exists k_ε such that, for all $k > k_\varepsilon$, we have

$$\inf_{n>n_K}\left\{n\log\left(\frac{K\Phi(n/T)}{N\left(\log\sqrt{r(k)}\right)}\right)\right\} \le -\frac{T(1-\varepsilon)}{(1+b)K^b}\log\sqrt{r(k)}.$$

Reporting this in (7.17) yields $|g_k| \le r(k)^{-\frac{T(1-\varepsilon)}{2(1+b)K^b}}$, for $k > k_\varepsilon$. By hypothesis, $|u_k| \le Cr(k)^a$. Hence

$$|g| = |\sum_k g_k u_k| \le C\left(\|g\|_2(1+k_\varepsilon)r(k_\varepsilon)^a + \sum_{k_\varepsilon+1}^{\infty} r(k)^{-\frac{T(1-\varepsilon)}{2(1+b)K^b}+a}\right).$$

Since $T > 2(1+a)(1+b)$ and $\varepsilon \in (0,1), K \in (1,\infty)$ are arbitrary, we can pick ε, K such that

$$-\frac{T(1-\varepsilon)}{2(1+b)K^b} + a < -1.$$

Since $r(k) \ge 1+k$, for such a choice of ε, K the series $\sum r(k)^{-\frac{T(1-\varepsilon)}{2(1+b)K^b}+a}$ converges and it follows that $g \in L^\infty(X,\mu)$. Theorem 7.1 is proved.

Let us illustrate the hypotheses made in Theorem 7.1, especially (7.13), by looking at the case when $X = G$ is a compact metrizable group equipped with its normalized Haar measure and $-A$ is the infinitesimal generator of a vaguely continuous convolution semigroup of symmetric probability measures $(\mu_t)_{t>0}$. Hence, $e^{-tA} : f \mapsto e^{-tA}f = f * \mu_t$, $t > 0$, is a self-adjoint semigroup of contractions on $L^2(G)$. By the Peter-Weyl theorem, representation theory provides us with a complete set of orthonormal eigenfunctions for A, formed of normalized matrix coefficients $d_\rho^{1/2}\rho_{i,j}$ where ρ runs over of all irreducible representations (modulo equivalence) and $d_\rho = \dim V_\rho$ is the dimension of the irreducible unitary representation (ρ, V_ρ). Note that the $\rho_{i,j}$ depends on A, namely, $\rho_{i,j}(x) = \langle \rho(x)e_i, e_j \rangle$ where $(e_i)_1^{d_\rho}$ is an orthonormal basis of V_ρ which diagonalizes $A_\rho : V_\rho \to V_\rho$ where $A_\rho v = A\rho(x)v|_{x=e}$. Because we assume that G is metrizable, this orthonormal system is countable.

Let us first consider the case when G is abelian. In this case, all the irreducible representations are of dimension 1 and the matrix coefficients (one for each irreducible representation class) are the characters γ of G which form the dual group $\widehat{G}$ of G. In particular, viewed as a function on G, any γ has modulus $|\gamma| \equiv 1$. Thus, in this case, (7.13) holds with $a = 0$ and Theorem 7.1 applies (Section 7.2 below generalizes this result to the case when G is a metrizable locally compact abelian group).

When G is not abelian (7.13) holds with $a = 1/2$. Indeed, for any eigenvalue $\lambda = \lambda_i$, there exists a representation ρ and a corresponding normalized matrix coefficient $u = d_\rho^{1/2}\rho_{i,j}$ such that u is a normalized eigenfunction for λ. Moreover, λ must have multiplicity at least d_ρ (because each irreducible representation appears with multiplicity equal to its dimension in the regular representation). As $|\rho_{k,l}(x)| \leq 1$, it follows that

$$\|u\|_\infty \leq d_\rho^{1/2} \leq r(i)^{1/2}.$$

If we further assume that for each $t > 0$, the measure μ_t is central, that is, satisfies $\mu_t(xV) = \mu_t(Vx)$ for any $x \in G$ and any Borel set V, then (7.13) holds with $a = 1/4$. Indeed, in this case, an eigenvalue appearing at an irreducible representaion ρ appears in fact with multiplicity at least d_ρ^2. Hence we have

$$\|u_i\|_\infty \leq d_\rho^{1/2} \leq r(i)^{1/4}.$$

7.2 Convolution semigroups on abelian groups

In what follows G is a locally compact metrizable abelian group and $\widehat{G}$ denotes its dual (which is a locally compact σ-compact abelian group). The reader unfamiliar with this setting can assume that $G = \widehat{G} = \mathbb{R}^d$. The references [11, 19, 23] provide detailed background. Both G and $\widehat{G}$ are equipped with their respective Haar measures dx and $d\gamma$ which are chosen in such a way that the Fourier transform

$$f \mapsto \hat{f} : \hat{f}(\gamma) = \int_G \overline{(x, \gamma)} f(x) dx$$

is a unitary map from $L^2(G, dx)$ to $L^2(\widehat{G}, d\gamma)$.

Let $(\mu_t)_{t>0}$ be a vaguely continuous convolution semigroup of symmetric probability measures on G (a measure ν is symmetric if $\nu(V) = \nu(-V)$ for any Borel set V). By [11, Theorem 8.3], we have

$$\widehat{\mu_t}(\gamma) = e^{-t\psi(\gamma)}, \quad \gamma \in \Gamma,$$

where ψ is a continuous negative definite function. Let $-A$ be the infinitesimal generator of the self-adjoint semigroup of contractions $f \mapsto \mu_t * f$ on $L^2(G, dx)$.

Theorem 7.4 *Let M, Φ be as in* Definition 2.1. *Assume Φ does not take the value $+\infty$ and M is $\mathcal{C}^1$. Assume also that $t \mapsto -tM'(t)$ is decreasing on $(0,\infty)$, tends to ∞ when t tends to 0, and that there exists $b > 0$ such that* (2.9) *holds. If, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le e^{M(t)}$ then, for any $T > 2(1+b)$, we have*

$$g \in \bigcap_0^\infty \mathrm{Dom}(A^n) \ \ \textit{and} \ \ \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{\Phi(n/T)} \le 1 \implies g \in L^\infty(G, dx).$$

Proof : The proof is similar to that of Theorem 7.1 with some significant technical adjustements. With the notation introduced above, the ultracontractivity of e^{-tA} implies that the measures μ_t, $t > 0$, have bounded continuous densities and we write $d\mu_t(x) = f_t(x)dx$. Then

$$\|e^{-tA}\|_{2\to\infty} = \|f_t\|_2, \ \ \widehat{f}_t = e^{-t\psi}.$$

Hence, we have

$$\|f_t\|_2^2 = \|e^{-t\psi}\|_2^2 \le e^{2M(t)}.$$

By the second structure theorem for LCA groups (e.g., [23, page 110]), $\widehat{G} = \mathbb{R}^d \times H$ where H contains a compact subgroup P such that H/P is discrete. Because we assume that G is metrizable, the dual group $\widehat{G}$ is σ-compact ([19, Theorem 4.2.7]) and it follows that H/P is countable. Hence there exists a countable set $\Gamma \subset \widehat{G}$ and a compact $U \subset \widehat{G}$ such that

$$\widehat{G} = \bigcup_{\gamma\in\Gamma} U_\gamma \tag{7.18}$$

where $U_\gamma = \gamma + U$ and $U_\gamma \cap U_{\gamma'}$ has measure 0 if $\gamma \ne \gamma'$. It is convenient to fix the normalization of the Haar measure on Γ so that U has measure 1.

Recall that the function ψ is continuous, satisfies $\psi(0) = 0$, $\psi(\xi) = \psi(-\xi)$ and has the property that $\sqrt{\psi}$ is subadditive (see, e.g., [11, Prop. 7.15]). It follows that there is a constant C such that, for any $\gamma \in \Gamma$ and any $\xi, \zeta \in U_\gamma$, $|\sqrt{\psi(\xi)} - \sqrt{\psi(\zeta)}| \le C$. Moreover, by the Riemann-Lebesgue lemma, $\lim_{\xi\to\infty} \psi(\xi) = \infty$. Hence, for any $\varepsilon > 0$, we have

$$\forall\, \xi \in U_\gamma, \ \ (1+\varepsilon)^{-1}\psi(\gamma) \le \psi(\xi) \le (1+\varepsilon)\psi(\gamma) \tag{7.19}$$

for all but finitely many U_γ. Let $\mathbf{Z}_\varepsilon$ be the subset of Γ such that (7.19) holds on U_γ when $\gamma \in \mathbf{Z}_\varepsilon$. Let $\gamma_1, \gamma_2 \ldots$ be an ordering of $\mathbf{Z}_\varepsilon$ such $\psi(\gamma_i) \le \psi(\gamma_{i+1})$. Let N be related to M by (2.7). We claim that

$$\forall\, k = 1, 2, \ldots, \ \ \ \psi(\gamma_k) \ge (1+\varepsilon)^{-1} N\left(\log\sqrt{k}\right). \tag{7.20}$$

Indeed, for any k,

$$k e^{-2t(1+\varepsilon)\psi(\gamma_k)} \le \sum_1^\infty e^{-2t(1+\varepsilon)\psi(\gamma_i)} \le \sum_1^\infty \int_{U_{\gamma_i}} e^{-2t\psi(\xi)} d\xi \le \|e^{-t\psi}\|_2^2 \le e^{2M(t)}$$

and thus

$$(1+\varepsilon)\psi(\gamma_k) \geq \sup_{t>0} \frac{1}{t}\left(\log\sqrt{k} - M(t)\right) = N\left(\log\sqrt{k}\right).$$

Now, let g be a function on G such that

$$g \in \bigcap_0^\infty \mathrm{Dom}(A^m) \text{ and } \limsup_{m\to\infty} \frac{\|A^m g\|_2^{1/m}}{\Phi(m/T)} \leq 1.$$

Then, there exists m_0 such that for all $m > m_0$,

$$((1+\varepsilon)^{-1}\psi(\gamma_k))^{2m} \int_{U_{\gamma_k}} |\hat{g}(\xi)|^2 d\xi \leq \int_{U_{\gamma_k}} |\hat{g}(\xi)|^2 |\psi(\xi)|^{2m} d\xi \leq ((1+\varepsilon)\Phi(m/T))^{2m}.$$

Thus

$$\int_{U_{\gamma_k}} |\hat{g}(\xi)|^2 d\xi \leq \exp\left(2m\log\left(\frac{(1+\varepsilon)^3\Phi(m/T)}{N(\log\sqrt{k})}\right)\right).$$

By Lemma 7.3, there exists k_ε such that for all $k > k_\varepsilon$, we have

$$\int_{U_{\gamma_k}} |\hat{g}(\xi)|^2 d\xi \leq k^{-\frac{T(1-\varepsilon)}{(1+b)(1+\varepsilon)^{3b}}}.$$

Let N_ε be the number of γ contained in $\Gamma \setminus \mathbf{Z}_\varepsilon$ and write

$$\begin{aligned}
|g| &\leq \int_{\widehat{G}} |\hat{g}(\xi)| d\xi = \sum_{\gamma\in\Gamma} \int_{U_\gamma} |\hat{g}(\xi)| d\xi \\
&\leq \sum_{\Gamma\setminus\mathbf{Z}_\varepsilon} \left(\int_{U_\gamma} |\hat{g}(\xi)|^2 d\xi\right)^{1/2} + \sum_{\mathbf{Z}_\varepsilon} \left(\int_{U_\gamma} |\hat{g}(\xi)|^2 d\xi\right)^{1/2} \\
&\leq \sum_{\Gamma\setminus\mathbf{Z}_\varepsilon} \left(\int_{U_\gamma} |\hat{g}(\xi)|^2 d\xi\right)^{1/2} + \sum_{k\leq k_\varepsilon} \left(\int_{U_{\gamma_k}} |\hat{g}(\xi)|^2 d\xi\right)^{1/2} + \sum_{k>k_\varepsilon} \left(\int_{U_{\gamma_k}} |\hat{g}(\xi)|^2 d\xi\right)^{1/2} \\
&\leq (N_\varepsilon + k_\varepsilon)^{1/2} \|g\|_2 + \sum_{k>k_\varepsilon} \left(\int_{U_{\gamma_k}} |\hat{g}(\xi)|^2 d\xi\right)^{1/2} \\
&\leq (N_\varepsilon + k_\varepsilon)^{1/2} \|g\|_2 + \sum_{k>k_\varepsilon} k^{-\frac{T(1-\varepsilon)}{2(1+b)(1+\varepsilon)^{3b}}}.
\end{aligned}$$

As $T > 2(1+b)$ and $\varepsilon > 0$ is arbitrary, we can pick $\varepsilon > 0$ such that $\frac{T(1-\varepsilon)}{2(1+b)(1+\varepsilon)^{3b}} > 1$. Then the series $\sum k^{-\frac{T(1-\varepsilon)}{2(1+b)(1+\varepsilon)^{3b}}}$ converges and it follows that $g \in L^\infty$ as desired.

7.3 The case $M(t) = g(\log(1+1/t)), g \in \mathbf{SR}_\lambda,\ \lambda > 1$.

In this section, we come back to the general setting of Section 7.1. Assume that $M(t) = g(\log(1+1/t)), g \in \mathbf{SR}_\lambda,\ \lambda > 1$. Straightforward computations show that there are constants $c_1, \dots, c_4 \in (0,\infty)$ such that, for all x large enough, we have

$$e^{c_1 (g')^{-1}(x)} \le \Phi(x) \le e^{c_2 (g')^{-1}(x)}$$

and

$$e^{c_3 g^{-1}(x)} \le N(x) \le e^{c_4 g^{-1}(x)}.$$

One can also show, as in Section 6.2, that $x^{-1} F \circ \Phi(x)$ tends to infinity so that Theorem 4.5 does not apply. However, for any fixed $\lambda > 1$ and $g \in \mathbf{SR}_\lambda$, under the hypotheses of Theorem 7.1 and assuming that $(e^{-tA})_{t>0}$ is sub-Markovian, or under the hypotheses of Theorem 7.4, we obtain the equivalence between the following three properties:

1. There exists $c_5 \in (0,\infty)$ such that, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \le e^{c_5 g(\log(1+1/t))}$;
2. There exists $c_6 \in (0,\infty)$ such that for any $u \in \bigcap_0^\infty \mathrm{Dom}(A^n)$ we have

$$\limsup_{n\to\infty} \frac{\|A^n u\|_2^{1/n}}{e^{c_6 (g')^{-1}(n)}} < \infty \implies u \in L^\infty;$$

3. There are constants $c_7, c_8 \in (0,\infty)$ such that the Nash type inequality

$$\forall\, f \in \mathrm{Dom}(A) \text{ with } \|f\|_1 \le 1,\ \ c_7 \|f\|_2^2 \exp(c_8 g^{-1}[\log(1+\|f\|_2)]) \le \langle Af, f\rangle + \|f\|_2^2.$$

   holds.

Compare with Section 6.2 and note that, in the more general setting of Section 4, we were not able to prove the equivalence between the embedding property (2) above and the other two properties (1) and (3).

8 Examples

In this last section, we present explicit examples of situations where the various type of ultracontractivity behaviors discussed earlier do occur.

8.1 The infinite dimensional torus

The results obtain in this paper apply nicely to symmetric Gaussian semigroups (i.e., Brownian motions) on the infinite dimensional torus $\mathbb{T}^\infty = \mathbb{R}^\infty/\mathbb{Z}^\infty$. The most expeditive way to introduce symmetric Gaussian semigroups on $\mathbb{T}^\infty$ is to use Fourier analysis. The dual group of $\mathbb{T}^\infty$ is $\mathbb{Z}^{(\infty)}$, that is, the set of all sequences with finitely many non-zero entries in $\mathbb{Z}$. A convolution semigroup of measures $(\mu_t)_{t>0}$ is symmetric Gaussian (non-degenerate) if the Fourier transform of μ_t is of the form

$$\widehat{\mu_t}(\theta) = \exp(-t\langle \mathcal{A}\theta, \theta\rangle), \quad \theta \in \mathbb{Z}^{(\infty)}$$

where $\mathcal{A} = (a_{i,j})$ is symmetric positive definite, that is, satisfies $a_{i,j} = a_{j,i}$ and

$$\langle \mathcal{A}\theta, \theta\rangle = \sum a_{i,j}\theta_i\theta_j > 0 \text{ for all } \theta \neq 0.$$

On smooth cylindrical functions, the infinitesimal generator is given by

$$-Af = \sum_{i,j} a_{i,j}\partial_i f \partial_j f.$$

The simplest case is when the matrix $\mathcal{A}$ is diagonal which we now use to provide very explicit examples. When $\mathcal{A}$ is diagonal, we set $a_i = a_{i,i}$ and $N_{\mathcal{A}}(s) = \#\{i : a_i \leq s\}$. Obviously, we can obtain functions $N = N_{\mathcal{A}}$ with almost any prescribed (non-decreasing) behavior by choosing appropriately the growth of the coefficients a_i. For instance, picking $a_i = \log_{(2)}(i)$ yields $\log_{(2)}(N(s)) \approx s$, $a_i = (1+i)^\alpha$ gives $N(s) \approx s^{1/\alpha}$ and $a_i = e^{e^i}$ gives $N(s) \approx \log_{(2)}(s)$. In the following statement, $\mu_t(0)$ denotes the value at $0 \in \mathbb{T}^\infty$ of the continuous density of the measure μ_t w.r.t. Haar measure, if μ_t admits a continuous density, and $\mu_t(0) = \infty$ otherwise. With this notation, we have

$$\mu_{2t}(0) = \|e^{-tA}\|^2_{2\to\infty}.$$

Theorem 8.1 *Let $(\mu_t)_{t>0}$ be as above. Assume that $\mathcal{A}$ is diagonal and $N_{\mathcal{A}}$ is regularly varying of index $\lambda \in [0,\infty]$. Then, we have*

$$\log \mu_t(0) \sim \begin{cases} \frac{1}{2}\int_0^{1/t} N_{\mathcal{A}}(x)\frac{dx}{x} & \text{if} \quad \lambda = 0 \\ c_\lambda N_{\mathcal{A}}(1/t) & \text{if} \quad \lambda \in (0,\infty) \\ 2\int_0^\infty e^{-xt} dN_{\mathcal{A}}(x) & \text{if} \quad \lambda = \infty. \end{cases}$$

See [3, Theorem 3.18]. For instance, picking $a_i = e^{e^i}$ yields $\log \mu_t(0) \sim \frac{1}{2}\log\frac{1}{t}\log_{(2)}\frac{1}{t}$; picking $a_i = (1+i)^\alpha$ yields $\log \mu_t(0) \sim c_\alpha t^{-1/\alpha}$; and picking $a_i = (\log(1+i))^\alpha$ with $\alpha \in (1,\infty)$ yields $\log_{(2)} \mu_t(0) \sim c_\alpha t^{-1/(\alpha-1)}$ as t tends to 0.

8.2 Symmetric Lévy generators on $\mathbb{R}$

On $\mathbb{R}$, consider an operator A of the form

$$Au(x) = -\frac{1}{2}\int_{-\infty}^{\infty} [u(x+y) - 2u(x) + u(x-y)]\, d\Pi(y)$$

where the Lévy measure Π is a symmetric Radon measure such that

$$\int \frac{|y|^2}{1+|y|^2} d\Pi(y) < \infty.$$

This operator, originally defined on smooth compactly supported functions, is extended minimally to a self-adjoint operator in $L^2(\mathbb{R})$. The operator $-A$ is the generator of a translation invariant Markov semigroup $H_t = e^{-tA}$. Let μ_t be the measure such that

$$H_t u = \mu_t * u.$$

This semigroup can be described using Fourier transform. Namely,

$$\widehat{H_t u}(\xi) = e^{-t\psi(\xi)}\widehat{u}(\xi)$$

where

$$\psi(\xi) = 2\int_0^\infty [1 - \cos \xi x] d\Pi(x).$$

The function ψ is a continuous negative definite function and $\widehat{\mu_t} = e^{-t\psi}$. See [12, 21]. The asymptotic behaviour of Π around 0 reflects into the growth of ψ at infinity, and ultimately in the behaviour of $\mu_t(0)$ for small t. See [4] for an explicit connection.

Classical Examples :

(1) Consider $d\Pi(x) = c_\alpha |x|^{-1-2\alpha}dx$, $\alpha \in (0,1)$. Then $\psi(\xi) = |\xi|^{2\alpha}$ and $A = \left(-\left(\frac{d}{dx}\right)^2\right)^\alpha$ and H_t is the symmetric α-stable semigroup.

(2) Consider $d\Pi(x) = |x|^{-1}e^{-|x|}dx$. Then $\psi(\xi) = 2\log(1+|\xi|^2)$, $A = 2\log\left[1 - \left(\frac{d}{dx}\right)^2\right]$ and H_t is the symmetric Γ-semigroup.

By a general result of Berg and Forst [11], the measure μ_t is absolutely continuous and has a continuous density (with respect to Lebesgue measure) if and only if its Fourier transform $\widehat{\mu_t}$ is in L^1. Moreover, if $\widehat{\mu_t} \in L^1$,

$$\|e^{-(t/2)A}\|^2_{2\to\infty} = \mu_t(0) = 2\int_0^\infty e^{-t\psi(\xi)}d\xi = 2\int_0^\infty e^{-ts}d\Psi(s) \tag{8.21}$$

where

$$\Psi(s) = |\{t > 0 : \psi(t) \le s\}|$$

and $x \mapsto \mu_t(x)$ denotes the density of μ_t. It follows that $(H_t)_{t>0}$ is ultracontractive if and only if, for all $t > 0$, $e^{-t\psi} \in L^1$. In this case, the density $x \mapsto \mu_t(x)$ is continuous.

We now describe explicit results relating the behavior of ψ and Ψ at infinity to the behavior of $\mu_t(0)$ at 0. We use the notation introduced before Proposition 6.2 concerning regular variation. The following result is a consequence of (8.21) and the Laplace transform techniques of [13, Th. 1.7.1, Th. 4.12.2, Th. 4.12.11(i)], i.e., Karamata's and Kohlbecker's theorems.

Theorem 8.2 *1. Let $\alpha \in (0,2)$ and $\psi_0 \in \mathbf{R}_\alpha$. The following properties are equivalent.*

(a) $\psi \sim \psi_0$ at infinity.

(b) $\Psi \sim \psi_0^{-1}$ *at infinity.*

(c) $\mu_t(0) \sim \Gamma(1+1/\alpha)\psi_0^{-1}(1/t)$ *at* 0.

2. *Let* $c > 0$, $\alpha > 1$ *and* $\theta \in \mathbf{R}_\alpha$. *Set* $\Theta(x) = \theta(x)/x$. *Then the following properties are equivalent.*

   (a) $\psi(x) \sim c^\alpha \theta(\log x)$ *at infinity.*

   (b) $\log \Psi(x) \sim c^{-1}\theta^{-1}(x)$ *at infinity.*

   (c) $\log \mu_t(0) \sim (\alpha - 1)(\alpha c)^{\alpha/(1-\alpha)}\Theta^{-1}(1/t)$ *at* 0.

To discuss further examples, it is convenient to consider the case where there exists an increasing function ω with $\omega(x) = o(x)$ at infinity and such that

$$\Psi(x) = \int_0^x e^{\omega(s)} ds. \tag{8.22}$$

In fact, using Polya's Theorem (e.g., [18, Th. 4.3.1]), for any ω as above there exists a symmetric convolution semigroup of probability measures $(\mu_t)_{t>0}$ such that $\widehat{\mu}_t(\xi) = \exp(-t\Psi^{-1}(|\xi|))$. As $\Psi^{-1}(\xi)/\log \xi$ tends to $+\infty$ at $+\infty$, it follows that μ_t admits a C^∞ density for all $t > 0$. Using the monotone density theorem [13, Th. 4.12.10], we derive from Theorem 8.2(2) that, in this context, for any $\psi_0 \in \mathbf{R}_\alpha$, $\alpha \in (1, \infty)$, there is equivalence between

$$\omega \sim \psi_0^{-1} \text{ at infinity}$$

and

$$\log \mu_t(0) \sim (\alpha - 1)\alpha^{\alpha/(1-\alpha)}\Omega^{-1}(1/t)$$

where $\Omega(x) = \psi_0(x)/x$.

We are now interested in the two limit cases $\alpha = \infty$ and $\alpha = 1$. We start with the case $\alpha = \infty$. To be more precise, we will consider the case where ω is slowly varying. The next theorem follows from [13, Th.4.12.12].

Theorem 8.3 *Let* Ψ *be defined by* (8.22) *with* ω *non-negative increasing with* $\omega(x) = o(x)$ *at infinity. Let* $x \mapsto \mu_t(x)$ *be the smooth density of the associated convolution semigroup. Let* $\psi_0, \psi_0^{\#}$ *be a pair of de Bruijn conjugate slowly varying functions (see [13, p.29 and Appendix 5]). Then the following properties are equivalent.*

1. $\omega \sim 1/\psi_0$ *at infinity.*

2. $\log t\mu_t(0) \sim \psi_0^{\#}(1/t)$ *at* 0.

Example (3) Let $\alpha > 0$. Referring to (8.22) and the associated Lévy semigroup, the following properties are equivalent.

1. $\omega(x) \sim c(\log x)^{\alpha}$ at infinity
2. $\log(t\mu_t(0)) \sim c(\log 1/t)^{\alpha}$ at 0.

Indeed, in this case we have $\psi_0(x) = c^{-1}(\log x)^{-\alpha}$ and, by [13, Th. A5.2], $\psi_0^{\#} = 1/\psi_0$.

Finally, we illustrate by an example what happens in the limit case where ω is regularly varying of index $\alpha = 1$. See [3] for the needed Laplace transform techniques.

Example (4) Let $\alpha > 0$ and recall the notation (6.11) for iterated logarithms. Referring to (8.22) and the associated Lévy semigroup, the following properties are equivalent.

1. $\omega(x) \sim x(\log_{(n)} x)^{-\alpha}$ at infinity
2. $\log_{(n+1)}(\mu_t(0)) \sim t^{-\alpha}$ at 0.

More generally, we have the following result, which proves that the ultracontractivity function of a Lévy semigroup can explode arbitrarily fast at zero. Recall a definition from [21, def.23.2, p.147]: a (probability) measure ρ on $\mathbb{R}$ is called unimodal with mode a if the functions

$$x \to \rho(]-\infty, x]), x \to \rho([x, +\infty[)$$

are convex on $]-\infty, a[$ and $]a, +\infty[$, respectively. That is,

$$\rho = c\delta_a + f(x)dx,$$

where $0 \leq c < \infty$ and f is increasing on $x < a$ and decreasing on $x > a$.

Theorem 8.4 *For any function G increasing to infinity at ∞, there exists a symmetric Lévy semigroup with a C^{∞} unimodal density $x \to \mu_t(x)$ such that*

$$\lim_{t \to 0^+} \frac{\mu_t(0)}{G(1/t)} = +\infty.$$

Proof : Let $\tilde{G}$ be any non-decreasing function from the class $\mathbf{R}_{\infty}$ such that $\tilde{G}/G \to \infty$ at ∞, e.g. one can take

$$\tilde{G}(x) = \exp\Big(\int_0^x G(s)ds\Big).$$

Define $\tilde{m} = \tilde{G}^{-1} \in \mathbf{R}_0$ and let $\tilde{\omega}(x) = x/\tilde{m}(x) \in \mathbf{R}_1$. By [13, Th.1.8.2], there exists $\omega_* \in \mathbf{SR}_1$ (smoothly varying function, see Sec. 6.5) such that ω_* is increasing, $\omega_* \geq \tilde{\omega}$, and $\omega_* \sim \tilde{\omega}$ at ∞. Define $f_* = \exp(\omega_*)$; then f_* is increasing, $f_* \in C^{\infty}$ and $1/f_*^2$ is eventually convex (at ∞). Indeed, convexity of $1/f_*^2$ at ∞ follows from the inequality

$$\omega_*'' - 2(\omega_*')^2 < 0 \qquad \text{at } \infty. \tag{8.23}$$

To prove this inequality we note that since $\omega_* \in \mathbf{SR}_1$, $x\omega_*'(x) \sim \omega_*(x)$ and $x^2\omega_*''(x)/\omega_*(x) \to 0$ at ∞ (see Sec.6.5). In particular, $\omega_*'' = o(\omega_*^2)$ at ∞ and the result follows.
Since $\omega_*(x) \sim \tilde{\omega}(x) = o(x)$ at ∞, $\omega_*'(x) \sim \omega_*(x)/x \to 0$ at ∞, also $\omega_*' \geq 0$. It follows that there exists $x_* >> 1$ such that $\omega_*''(x_*) < 0$ and the inequality (8.23) holds for $x \geq x_*$. Define an increasing smooth function ω such that $\omega(x) = \omega_*(x)$ for $x \geq x_*$ and $\omega(x)$ is concave for $0 < x < x_*$. Then ω satisfies the inequality (8.23) for all $x > 0$. Hence the function $f = \exp(\omega)$ is increasing to infinity at infinity and the function $1/f^2$ is convexe. Let Ψ be defined by (8.22) with ω as above. Let $(\mu_t)_{t>0}$ be the symmetric convolution semigroup of probability measures such that $\widehat{\mu}_t(\xi) = \exp(-t\Psi^{-1}(|\xi|))$. Let $x \to \mu_t(x)$ be the corresponding smooth density. By Askey's Theorem [1], the density $x \to \mu_t(x)$ is unimodal with mode 0 if its characteristic function $\Phi_t : \xi \to \exp(-t\Psi^{-1}(|\xi|))$ satisfies the following condition: $\xi \to -\Phi_t'(\xi)$ is convex on the interval $\{\xi > 0\}$. That this condition holds follows easily from the convexity of the function $1/f^2$. Further we have

$$\begin{aligned}
\mu_t(0) &= 2\int_0^\infty e^{-st} f(s)ds \geq \int_{x_*}^\infty e^{-st} f_*(s)ds \\
&= \int_{x_*}^\infty e^{-st} e^{\omega_*(s)}ds \geq \int_{x_*}^\infty e^{-st+\tilde{\omega}(s)}ds \\
&= \int_{x_*}^\infty e^{-st+s/\tilde{m}(s)}ds = \int_{x_*}^\infty e^{s(1/\tilde{m}(s)-t)}ds.
\end{aligned}$$

Fot t small enough, define s_0 as the unique solution of the equation $\tilde{m}(s_0) = 1/t$. Since $\tilde{m}$ increases to infinity at infinity, $s_0 \to \infty$ as $t \to 0$. Therefore for $0 < t < T$ small enough and such that $s_0 = s_0(t) > x_*$, we obtain

$$\begin{aligned}
\mu_t(0) &\geq \int_{x_*}^{s_0} e^{s(1/\tilde{m}(s)-t)}ds \geq s_0 - x_* \\
&\sim \tilde{m}^{-1}(1/t) = \tilde{G}(1/t), \quad t \to 0.
\end{aligned}$$

Hence, because $\tilde{G}/G \to \infty$ at ∞,

$$\lim_{t\to 0} \mu_t(0)/G(1/t) = \infty.$$

The proof is finished.

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