Riesz Transform and Related Inequalities on Noncompact Riemannian Manifolds

THIERRY COULHON
Université de Cergy-Pontoise
AND
XUAN THINH DUONG
Macquarie University

1 Introduction

Let M be a complete, noncompact Riemannian manifold, μ the Riemannian measure, ∇ the Riemannian gradient, and Δ the (positive) Laplace-Beltrami operator on M. Denote by $|\cdot|$ the length in the tangent space, and by $\|\cdot\|_p$ the norm in $L^p(M,\mu)$, $1 \le p \le +\infty$.

If one wants to define homogeneous Sobolev spaces of order one on M, that is, spaces of functions with one derivative in $L^p(M,\mu)$, $1 < p < +\infty$, there are two obvious candidates for the seminorm: $\||\nabla f|\|_p$ and $\|\Delta^{1/2} f\|_p$. The former is local and of geometric nature, the latter is nonlocal and more analytic. When M is the Euclidean space, these two seminorms are equivalent for all $p \in]1,+\infty[$:

$$C_p^{-1}\|\Delta^{1/2} f\|_p \le \||\nabla f|\|_p \le C_p\|\Delta^{1/2} f\|_p \quad \forall f \in \mathcal{C}_0^\infty(\mathbb{R}^n)\,.$$

This relies on singular integral theory (see [40, 42]); indeed, the second inequality above is nothing but the L^p-boundedness of the so-called Riesz transforms $\frac{\partial}{\partial x_i}\Delta^{-1/2}$, $i = 1,\dots,n$ in $\mathbb{R}^n$, and the first one follows by duality.

It is, of course, a basic issue (which was raised in [43]) to ask for which complete, noncompact Riemannian manifolds M and which $p \in]1,+\infty[$ one has

$$C_p^{-1}\|\Delta^{1/2} f\|_p \le \||\nabla f|\|_p \le C_p\|\Delta^{1/2} f\|_p \quad \forall f \in \mathcal{C}_0^\infty(M). \tag{1.1}$$

For $p = 2$, on any complete Riemannian manifold, one has the equality

$$\||\nabla f|\|_2 = \|\Delta^{1/2} f\|_2\,, \tag{1.2}$$

and this may even be used to define the Laplace-Beltrami operator. For $p \ne 2$, the above equivalence of seminorms is by no means a trivial matter, and a lot of work has been devoted to its proof for several classes of manifolds; see, for instance, [1, 3, 4, 5, 31, 33].

A typical result, due to Bakry, is that (1.1) holds on any complete Riemannian manifold with nonnegative Ricci curvature for all $p \in]1,+\infty[$. The drawback of this beautiful result is that the nonnegativity of Ricci curvature can be destroyed

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very easily by local perturbations of the manifold or the metric, whereas one expects (1.1) to be somewhat more stable.

In [14], we proved that the Riesz transform $\nabla\Delta^{-1/2}$ is of weak type $(1,1)$ and therefore bounded on $L^p(M,\mu)$ for $1 < p < 2$, assuming that M satisfies the doubling property and a heat kernel upper estimate; note that these sufficient conditions are invariant under quasi-isometry. For variations on the proof as well as interesting generalizations, see [7, 39]; for extensions to vector bundles, see [44]; and for an H^1 theory, see [34]. We also gave in [14] an example showing that, for $p > 2$, stronger assumptions are needed.

Denote by $H_t = e^{-t\Delta}$ the heat semigroup on M, and consider the following property:

$$|\nabla H_t f| \le C H_t|\nabla f| \quad \forall f \in \mathcal{C}_0^\infty(M),\ \forall t > 0. \tag{1.3}$$

In some sense, (1.3) says in a strong way that the heat semigroup preserves the Lipschitz functions, since it implies

$$\||\nabla H_t f|\|_\infty \le C\||\nabla f|\|_\infty \quad \forall f \in \mathcal{C}_0^\infty(M)\,.$$

It can also be seen as a weak form of semigroup domination, if one calls semigroup domination (see [28]) the condition

$$|\vec{H}_t\omega| \le C H_t|\omega| \quad \forall \omega \in \mathcal{C}^\infty(M, T^*M)\,. \tag{1.4}$$

Here $\vec{H}_t$ stands for the heat semigroup on 1-forms, and the notation will be explained in Section 5.1 below. In fact, (1.3) is nothing but (1.4) restricted to exact forms. For more on these conditions, see Section 5.2. Note that in the literature on semigroup domination, starting with [28], the constant C in (1.4) is 1 or is replaced by an exponential e^{ct}. As a matter of fact, condition (1.4), and therefore (1.3), holds with $C = 1$ on manifolds with nonnegative Ricci curvature. More precisely, (1.3) with $C = 1$ is *equivalent* to the nonnegativity of the Ricci curvature (see [3, 4]). It is important for us to consider arbitrary values of C, which means that we are not bound to a curvature assumption.

Condition (1.3) implies

$$|\nabla H_t f|^2 \le C' H_t(|\nabla f|^2) \quad \forall f \in \mathcal{C}_0^\infty(M),\ \forall t > 0, \tag{1.5}$$

with $C' = C^2$. Indeed, since H_t is a sub-Markovian operator,

$$(H_t|\nabla f|)^2 \le H_t(|\nabla f|^2)\,.$$

One can find in [32, theorem 3.3] a sufficient condition for (1.5) to hold, more general than the nonnegativity of the Ricci curvature. Instead of (1.5), we could deal at the expense of slight modifications of the proofs with conditions such as

$$|\nabla H_t f|^2 \le C H_{\theta t}(|\nabla H_{\eta t} f|^2) \quad \forall f \in \mathcal{C}_0^\infty(M),\ \forall t > 0,$$

with $\theta, \eta > 0$, but for simplicity we will mostly stick to (1.5).

We formulate the following.

CONJECTURE 1.1 *Let M be a complete Riemannian manifold. There exists C_p such that*

$$\||\nabla f|\|_p \leq C_p \|\Delta^{1/2} f\|_p \quad \forall f \in \mathcal{C}_0^\infty(M) \tag{1.6}$$

if

- $1 < p \leq 2$

or

- $2 < p < +\infty$ *and* (1.5) *holds.*

So far we were only able to prove a weak version of (1.6) under the above assumptions, namely replacing $\|\Delta^{1/2} f\|_p$ by the larger quantity $\sqrt{\|f\|_p \|\Delta f\|_p}$ (see Sections 2.2 and 4 below), and to prove (1.6) under stronger assumptions, namely the doubling property and a suitable estimate on the heat kernel on functions if $1 < p < 2$ (see [14]), together with the domination condition (1.4) instead of (1.5) if $p > 2$ (see Section 5.2 below). In Section 6, we also study a consequence of (1.6) for $p > 2$, namely, the reverse inequality

$$\|\Delta^{1/2} f\|_p \leq C_p \||\nabla f|\|_p \quad \forall f \in \mathcal{C}_0^\infty(M) \text{ for } 1 < p < 2.$$

The proofs of some of the results of the present paper were sketched in [15]. The contents of Sections 2, 3, and 4 are valid in the abstract Markov semigroup and "carré du champ" setting of [3].

The essential ingredients in our proofs are, on the one hand, the Duong-McIntosh singular integral theory [2, 24] and, on the other hand, Paul-André Meyer's probabilistic Littlewood-Paley theory [35, 36]. We use only the basic form of the latter; it might be the case that progress on the above conjecture depends on the use of more sophisticated Littlewood-Paley inequalities as in [3] (but, of course, without the nonnegative Ricci curvature assumption).

The reader may wonder how large the class of manifolds is that we can treat with our methods. As we already mentioned, a sufficient condition for (1.5) to hold (as well as a sufficient condition for a slightly weaker condition than (1.4)) can be found in [32]. These conditions are in terms of estimates on some Schrödinger semigroups, where a lower bound on the Ricci curvature plays the role of a potential. As one can expect, the general idea is that if the Ricci curvature is not too negative, then one is in business, although this is far from being explicit in computational terms. We feel that a lot of work still has to be done in order to understand properly (1.4) and (1.5). For instance, do they hold on a cocompact covering with polynomial growth? For more considerations on the scope of our results, see Section 5.2 below.

2 Preliminary Considerations

2.1 Dualizing the Inequalities

This section is about the relationship between the two inequalities in (1.1). It is well-known that

$$\||\nabla f|\|_p \le C_p \|\Delta^{1/2} f\|_p \quad \forall f \in \mathcal{C}_0^\infty(M)$$

implies

$$\|\Delta^{1/2} f\|_q \le C_p \||\nabla f|\|_q \quad \forall f \in \mathcal{C}_0^\infty(M)$$

where q is the conjugate exponent to p, and it is well-known that the converse is not clear.

Explaining in detail this phenomenon will demonstrate the role of 1-forms in the Riesz transform problem.

Let $\mathcal{C}^\infty(M; T^*M)$ be the space of smooth 1-forms, and $L^p(M; T^*M)$ the space of 1-forms in L^p (for simplicity we shall omit the measure μ). Denote by $|\cdot|$ the length in the tangent space T_xM or in the cotangent space T_x^*M, $x \in M$, according to the context; by extension, for ω a 1-form, we shall denote by $|\omega|$ the function $x \mapsto |\omega(x)|$ on M, and, for X a vector field, by $|X|$ the function $x \mapsto |X(x)|$. With this notation, if $f \in \mathcal{C}^\infty(M)$, $df \in \mathcal{C}^\infty(M; T^*M)$, and $|df| = |\nabla f|$. We shall denote by $u \cdot v$ the scalar product in T_xM or T_x^*M, and by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(M, \mu)$ or $L^2(M; T^*M)$.

For $p \in [1, +\infty[$, define

$$D_p = \overline{\{df : f \in \mathcal{C}_0^\infty(M)\}},$$

where the closure is taken in $L^p(M; T^*M)$; in short, D_p is the space of exact forms in L^p.

The following proposition is implicit in [4].

PROPOSITION 2.1 *Let M be a complete Riemannian manifold. Let $1 < p < +\infty$, and let q be the conjugate exponent to p. If*

$$\||\nabla f|\|_p \le C_p \|\Delta^{1/2} f\|_p \quad \forall f \in \mathcal{C}_0^\infty(M), \tag{2.1}$$

then

$$\|\Delta^{1/2} f\|_q \le C'_p \||\nabla f|\|_q \quad \forall f \in \mathcal{C}_0^\infty(M). \tag{2.2}$$

Conversely, if

$$\||df|\|_p \le C \sup\left\{\langle df, dg\rangle : g \in \mathcal{C}_0^\infty(M), \||dg|\|_q \le 1\right\}, \tag{2.3}$$

that is, in short, if $D_p^* = D_q$, then (2.2) *implies* (2.1).

PROOF: Let $f, g \in \mathcal{C}_0^\infty(M)$. The very construction of the gradient implies that

$$\langle df, dg\rangle = \int_M \nabla f(x) \cdot \nabla g(x) d\mu(x).$$

Now, the polarized version of (1.2) yields

$$\int_M \nabla f(x) \cdot \nabla g(x) d\mu(x) = \langle \Delta f, g \rangle ,$$

and, from the definition of the square root of the Laplace operator,

$$\langle \Delta f, g \rangle = \langle \Delta^{1/2} f, \Delta^{1/2} g \rangle .$$

Finally, for $f, g \in \mathcal{C}_0^\infty(M)$,

$$\langle df, dg \rangle = \int_M \nabla f(x) \cdot \nabla g(x) d\mu(x) = \langle \Delta^{1/2} f, \Delta^{1/2} g \rangle . \tag{2.4}$$

It follows from (2.4) that

$$\langle \Delta^{1/2} f, \Delta^{1/2} g \rangle \leq \int_M |\nabla f(x)||\nabla g(x)| d\mu(x) \leq \| |\nabla f| \|_q \| |\nabla g| \|_p ;$$

therefore, if (2.1) holds,

$$\langle \Delta^{1/2} f, \Delta^{1/2} g \rangle \leq C_p \| |\nabla f| \|_q \| \Delta^{1/2} g \|_p . \tag{2.5}$$

Now

$$\| \Delta^{1/2} f \|_q = \sup \left\{ \langle \Delta^{1/2} f, \Delta^{1/2} g \rangle : g \in \mathcal{C}_0^\infty(M), \| \Delta^{1/2} g \|_p \leq 1 \right\}$$

because $\Delta^{1/2} \mathcal{C}_0^\infty$ is dense in L^p (this is classical; see, for instance, the appendix in [37] for a nice presentation). Together with (2.5), this yields (2.2).

Conversely,

$$\| |\nabla f| \|_p = \| |df| \|_p = \sup \left\{ \langle df, \omega \rangle : \omega \in L^q(M; T^*M), \| |\omega| \|_q \leq 1 \right\} ,$$

but if one has the better estimate (2.3) one writes, again thanks to (2.4),

$$\langle df, dg \rangle \leq \| \Delta^{1/2} f \|_p \| \Delta^{1/2} g \|_q ,$$

and (2.1) follows from (2.2). □

Note that the validity of (2.3) is connected with the validity of the Hodge decomposition in $L^p(M; T^*M)$; this condition holds if the L^p norm of the exact part of any smooth 1-form in the smooth Hodge decomposition is comparable with the L^p norm of the form itself. This is certainly not true in general on noncompact Riemannian manifolds, and the problem of finding sufficient conditions is not easier than the Riesz transform problem (see some related remarks in [32, intro.]).

2.2 Interpolation Inequalities

The following inequalities are well-known, although not so easy to find in the literature; see, for instance, [19, app. 2] and the references therein. The case $p = q = r$ is more standard; see [30, prop. 5.5].

PROPOSITION 2.2 *Let M be a complete Riemannian manifold and let $p, q, r \in]1, +\infty[$ such that $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$. Then*

$$\|\Delta^{1/2} f\|_p^2 \leq C_p \|f\|_q \|\Delta f\|_r \quad \forall f \in \mathcal{C}_0^\infty(M)\,.$$

It follows from the case $p = q = r$ of the above proposition that Conjecture 1.1 is stronger than Theorems 4.1 and 4.3 below.

3 Vertical Littlewood-Paley-Stein Function

Let $P_t = e^{-t\sqrt{\Delta}}$ be the Poisson semigroup on M. The vertical Littlewood-Paley function of a function f on M is defined by

$$\mathcal{G}(f)(x) = \left(\int_0^{+\infty} |\nabla P_t f(x)|^2 t\, dt\right)^{\frac{1}{2}}\,.$$

If $p \in]1, +\infty[$, one says that $\mathcal{G}$ is bounded on $L^p(M, \mu)$ if

$$\left\| \left(\int_0^{+\infty} |\nabla P_t f(\,\cdot\,)|^2 t\, dt\right)^{\frac{1}{2}} \right\|_p \leq C_p \|f\|_p \quad \forall f \in \mathcal{C}_0^\infty(M)\,.$$

The argument in [41, pp. 50–51] shows that, if M is the Euclidean space, $\mathcal{G}$ is bounded on L^p for all $p \in]1, 2]$; this argument extends verbatim to a complete Riemannian manifold (see [16]).

For $p > 2$, using martingale methods, which work in a general Markov diffusion semigroup setting, Paul-André Meyer was able to prove another kind of Littlewood-Paley estimate (see [35, 36]; one can find the elements of an analytic proof in [41, pp. 52–54]):

$$\left\| \left(\int_0^{+\infty} P_t\left(|\nabla P_t f|^2\right)(\,\cdot\,) t\, dt\right)^{\frac{1}{2}} \right\|_p \leq C_p \|f\|_p \quad \forall f \in \mathcal{C}_0^\infty(M)\,. \tag{3.1}$$

Assume now that (1.5) holds. Since the heat semigroup and the Poisson semigroup are related by the subordination formula

$$P_t = \int_0^{+\infty} e^{-u} u^{-1/2} H_{t^2/4u}\, du\,,$$

it follows easily that

$$|\nabla P_t f|^2 \leq C P_t(|\nabla f|^2) \quad \forall f \in \mathcal{C}_0^\infty(M),\ \forall t > 0.$$

Therefore, according to (3.1), $\mathcal{G}$ is bounded on $L^p(M, \mu)$.

To summarize, we state the following.

PROPOSITION 3.1 *Let M be a complete Riemannian manifold. Then $\mathcal{G}$ is bounded on $L^p(M,\mu)$ provided*

- $1 < p \leq 2$

or

- $2 < p < +\infty$ *and* (1.5) *holds.*

Proposition 3.1 is implicit in [32]. Together with the criterion for (1.5) in [32, theorem 4.2], it yields directly [32, theorem 5.1].

4 Multiplicative Inequalities

THEOREM 4.1 *Let M be a complete Riemannian manifold. There exists C_p such that*

$$\||\nabla f|\|_p^2 \leq C_p \| f \|_p \| \Delta f \|_p \quad \forall f \in \mathcal{C}_0^\infty(M) \tag{4.1}$$

if

- $1 < p \leq 2$

or

- $2 < p < +\infty$ *and* (1.5) *holds.*

REMARKS

- The above statement does not put any constraint on the volume growth, such as the doubling property. In other words, it is a dimension-free statement.
- The same proof as in [14, sec. 5] shows that there are very simple manifolds (e.g., two copies of the plane glued along a circle) where (4.1) does not hold for $p > 2$ (it follows that (1.5) fails on such manifolds).
- It is easy to deduce from the above statement that, on any complete Riemannian manifold, the only L^p-harmonic functions, for $1 < p \leq 2$, are constants. This is in fact true for all $1 < p < +\infty$ as shown by Yau ([45]).

Theorem 4.1 follows from Proposition 3.1 and the following.

PROPOSITION 4.2 *Let M be a complete Riemannian manifold and $p \in]1, +\infty[$. Assume that $\mathcal{G}$ is bounded on $L^p(M,\mu)$. Then there exists C_p such that*

$$\||\nabla f|\|_p^2 \leq C_p \| f \|_p \| \Delta f \|_p \quad \forall f \in \mathcal{C}_0^\infty(M) \, .$$

PROOF: Let $f \in \mathcal{C}_0^\infty(M)$. Because of the L^2 Hodge decomposition (see [21, theorem 24, p. 165]), the differential df, being exact, has no harmonic part; therefore $\nabla P_t f = \vec{P}_t \, df$ tends to zero in L^2 as t goes to infinity (we owe this argument to Anton Thalmaier). Since f is smooth, $\nabla P_t f$ tends pointwise to ∇f as t goes to zero; thus one may write

$$|\nabla f|^2 = - \int_0^{+\infty} \frac{d}{dt} |\nabla P_t f|^2 \, dt \, .$$

Now

$$-\int_0^{+\infty} \frac{d}{dt}|\nabla P_t f|^2\,dt = -\Big[t\frac{d}{dt}|\nabla P_t f|^2\Big]_0^{+\infty} + \int_0^{+\infty} \frac{d^2}{dt^2}|\nabla P_t f|^2 t\,dt$$
$$\leq \int_0^{+\infty} \frac{d^2}{dt^2}|\nabla P_t f|^2 t\,dt\,.$$

Thus

$$\begin{aligned}|\nabla f|^2 &\leq \int_0^{+\infty} \frac{d^2}{dt^2}|\nabla P_t f|^2 t\,dt\\ &= -2\int_0^{+\infty} \frac{d}{dt}\left\{\nabla\Delta^{1/2}P_t f\cdot\nabla P_t f\right\} t\,dt\\ &= 2\int_0^{+\infty} |\nabla\Delta^{1/2}P_t f|^2 t\,dt + 2\int_0^{+\infty}\{\nabla\Delta P_t f\cdot\nabla P_t f\}\,t\,dt\\ &= 2(\mathrm{I}+\mathrm{II})\,.\end{aligned}$$

If $\mathcal{G}$ is L^p-bounded,

$$\|\mathrm{I}\|_{p/2} = \|\mathcal{G}(\Delta^{1/2}f)\|_p^2 \leq C\|\Delta^{1/2}f\|_p^2 \leq C'\|f\|_p\|\Delta f\|_p\,,$$

according to Proposition 2.2. As for II, write

$$\begin{aligned}|\mathrm{II}| &\leq \int_0^{+\infty} |\nabla\Delta P_t f||\nabla P_t f|t\,dt\\ &\leq \left(\int_0^{+\infty}|\nabla\Delta P_t f|^2 t\,dt\right)^{\frac12}\left(\int_0^{+\infty}|\nabla P_t f|^2 t\,dt\right)^{\frac12} = \mathcal{G}(\Delta f)\mathcal{G}(f)\,.\end{aligned}$$

Hence by Cauchy-Schwarz,

$$\|\mathrm{II}\|_{p/2} \leq \|\mathcal{G}(\Delta f)\|_p\|\mathcal{G}(f)\|_p\,;$$

therefore, again by the L^p-boundedness of $\mathcal{G}$,

$$\|\mathrm{II}\|_{p/2} \leq C\|\Delta f\|_p\|f\|_p\,,$$

and the proposition is proven. □

Replacing Cauchy-Schwarz in the proof of Proposition 4.2 by a more general Hölder inequality, one obtains the following theorem in the same way.

THEOREM 4.3 *Let M be a complete Riemannian manifold and $p,q,r\in\,]1,+\infty[$ be such that $\frac{2}{p}=\frac1q+\frac1r$. There exists $C_{p,q,r}$ such that*

$$\||\nabla f|\|_p^2 \leq C_{p,q,r}\|f\|_q\|\Delta f\|_r \quad \forall f\in\mathcal{C}_0^\infty(M)$$

if

- $\max\{q,r\}\leq 2$

or

- $\max\{q,r\}>2$ *and* (1.5) *holds.*

5 Riesz Transform

The connection between the Riesz transform, Littlewood-Paley theory, and the heat semigroup on 1-forms was first pointed out in [4] (see also [22]). We now enter this circle of ideas and try to extend it beyond the framework of nonnegative Ricci curvature.

5.1 Riesz Transform and Littlewood-Paley Functions

For $f \in \mathcal{C}_0^\infty(M)$, denote by $\|f\|_p$ the L^p norm of f with respect to $d\mu$. According to the context, $|\cdot| = |\cdot|_x$ will denote the norm in T_xM or in T_x^*M.

Denote by $L^2(M;\Lambda^k T^*M)$ the space of k-forms in L^2 equipped with its scalar product $\langle\cdot,\cdot\rangle$, by $d^k : L^2(M;\Lambda^k T^*M) \mapsto L^2(M;\Lambda^{k+1}T^*M)$ the exterior differentiation operator, and by $\delta^k : L^2(M;\Lambda^{k+1}T^*M) \mapsto L^2(M;\Lambda^k T^*M)$ its adjoint. Denote by $\vec{\Delta}$ the Hodge–de Rham Laplacian $d^0\delta^0 + \delta^1 d^1$, which acts as an unbounded self-adjoint operator on $L^2(M;T^*M) = L^2(M;\Lambda^1 T^*M)$.

Let $\vec{H}_t = e^{-t\vec{\Delta}}$ be the heat semigroup on 1-forms, a semigroup of self-adjoint operators acting on $L^2(M;T^*M)$.

The basic relation between $\vec{H}_t$ and the heat semigroup on functions $H_t = e^{-t\Delta}$ is that they are intertwined by d:

$$\vec{H}_t\, d = d H_t\,. \tag{5.1}$$

For a nice account of this, see [23, sec. 9.2].

Denote by $\vec{P}_t = e^{-t\sqrt{\vec{\Delta}}}$ the Poisson semigroup on 1-forms also acting on $L^2(M;T^*M)$. Since $\vec{P}_t$ and $\vec{H}_t$ are related by the subordination formula

$$\vec{P}_t = \int_0^{+\infty} e^{-u} u^{-1/2} \vec{H}_{t^2/4u}\, du\,,$$

it follows from (5.1) that

$$\vec{P}_t\, d = d P_t\,. \tag{5.2}$$

Define the horizontal Littlewood-Paley-Stein $\vec{g}$-function on 1-forms by

$$\vec{g}(\omega)(x) = \left(\int_0^{+\infty} \left|\frac{\partial}{\partial t}\vec{P}_t\omega(x)\right|^2 t\, dt\right)^{\frac{1}{2}} \quad \text{for } \omega \in L^2(M;T^*M),\ x \in M.$$

One says that $\vec{g}$ is bounded on $L^p(M;T^*M)$ if

$$\|\vec{g}(\omega)\|_p \le \|\omega\|_p \quad \forall \omega \in L^p(M;T^*M)\,.$$

Recall that

$$\mathcal{G}(f)(x) = \left(\int_0^{+\infty} |\nabla P_t f(x)|^2 t\, dt\right)^{\frac{1}{2}} \quad \text{for, say, } f \in L^2(M),\ x \in M.$$

Here is our key argument, which was already presented in [15]. It is a simple version of some of the original arguments of Bakry; see [4, 5]. We were also inspired by [22] and [9, p. 381].

THEOREM 5.1 *Let M be a complete Riemannian manifold and $p \in]1,+\infty[$. Assume that $\mathcal{G}$ is bounded on $L^p(M,\mu)$ and that $\vec{g}$ is bounded on $L^q(M;T^*M)$, $\frac{1}{p}+\frac{1}{q}=1$. Then the Riesz transform $\nabla\Delta^{-1/2}$ is bounded on $L^p(M,\mu)$; that is, there exists C_p such that*

$$\||\nabla f|\|_p \le C_p \|\Delta^{1/2} f\|_p \quad \forall f \in \mathcal{C}_0^\infty(M)\,.$$

PROOF: Let $p \in]1,+\infty[$ and $q=\frac{p}{p-1}$. For $f \in \mathcal{C}_0^\infty(M)$, write

$$\||\nabla\Delta^{-1/2} f|\|_p = \||d\Delta^{-1/2} f|\|_p = \sup_{\|\omega\|_q\le 1} |\langle \omega, d\Delta^{-1/2} f\rangle|\,.$$

Since $d\Delta^{-1/2} f$ is not harmonic (see the argument at the beginning of the proof of Proposition 4.2),

$$d\Delta^{-1/2} f = 4\int_0^{+\infty}\left(\frac{\partial^2}{\partial t^2}\vec{P}_{2t}\, d\Delta^{-1/2} f\right) t\, dt$$

and

$$\langle \omega, d\Delta^{-1/2} f\rangle = 4\int_0^{+\infty}\left\langle \frac{\partial}{\partial t}\vec{P}_t\omega, \frac{\partial}{\partial t}\vec{P}_t\, d\Delta^{-1/2} f\right\rangle t\, dt\,.$$

Therefore

$$\||\nabla\Delta^{-1/2} f|\|_p \le 4 \sup_{\|\omega\|_q\le 1}\left|\int_0^{+\infty}\left\langle \frac{\partial}{\partial t}\vec{P}_t\omega, \frac{\partial}{\partial t}\vec{P}_t\, d\Delta^{-1/2} f\right\rangle t\, dt\right|.$$

Now, thanks to (5.2), one has

$$\frac{\partial}{\partial t}\vec{P}_t\, d\Delta^{-1/2} = d\frac{\partial}{\partial t}P_t\Delta^{-1/2} = -dP_t\,,$$

thus

$$\begin{aligned}
\||\nabla\Delta^{-1/2} f|\|_p &= 4 \sup_{\|\omega\|_q\le 1}\left|\int_0^{+\infty}\left\langle \frac{\partial}{\partial t}\vec{P}_t\omega, dP_t f\right\rangle t\, dt\right|,\\
&= 4 \sup_{\|\omega\|_q\le 1}\left|\int_0^{+\infty}\int_M \frac{\partial}{\partial t}\vec{P}_t\omega(x)\cdot dP_t f(x) d\mu(x) t\, dt\right|\\
&\le 4 \sup_{\|\omega\|_q\le 1}\int_0^{+\infty}\int_M \left|\frac{\partial}{\partial t}\vec{P}_t\omega(x)\right| |dP_t f(x)| t\, dt\, d\mu(x)\\
&= 4 \sup_{\|\omega\|_q\le 1}\int_M\int_0^{+\infty} \left|\frac{\partial}{\partial t}\vec{P}_t\omega(x)\right| |\nabla P_t f(x)| t\, dt\, d\mu(x)
\end{aligned}$$

$$\leq 4 \sup_{\|\omega\|_q \leq 1} \int_M \left(\int_0^{+\infty} \left| \frac{\partial}{\partial t} \vec{P}_t \omega(x) \right|^2 t \, dt \right)^{\frac{1}{2}} \left(\int_0^{+\infty} |\nabla P_t f(x)|^2 t \, dt \right)^{\frac{1}{2}} d\mu(x)$$

$$\leq 4 \sup_{\|\omega\|_q \leq 1} \left\| \left(\int_0^{+\infty} \left| \frac{\partial}{\partial t} \vec{P}_t \omega(\cdot) \right|^2 t \, dt \right)^{\frac{1}{2}} \right\|_q \left\| \left(\int_0^{+\infty} |\nabla P_t f(\cdot)|^2 t \, dt \right)^{\frac{1}{2}} \right\|_p$$

$$= 4 \sup_{\|\omega\|_q \leq 1} \|\vec{g}(\omega)\|_q \|\mathcal{G}(f)\|_p \, .$$

Finally, the assumptions on $\mathcal{G}$ and $\vec{g}$ yield

$$\| |\nabla \Delta^{-1/2} f| \|_p \leq C \| f \|_p \, .$$

□

We have already explained how one could obtain the boundedness of $\mathcal{G}$; as far as $\vec{g}$ is concerned, say for $2 < p < +\infty$, that is $1 < q < 2$, one can find sufficient conditions in [32, theorem 6.1].

For technical reasons, we are now going to give a slightly more sophisticated version of Theorem 5.1, which involves the Poisson semigroup on functions and the heat semigroup on 1-forms. The aim is to reduce the L^p-boundedness of Riesz transforms to the L^p-boundedness of $\mathcal{G}$ and the L^q-boundedness of the horizontal Littlewood-Paley function on 1-forms built with the heat semigroup

$$\vec{h}(\omega)(x) = \left(\int_0^{+\infty} \left| \frac{\partial}{\partial t} \vec{H}_t \omega(x) \right|^2 t \, dt \right)^{\frac{1}{2}} ,$$

and then to use estimates on $\vec{H}_t$ to deal with $\vec{h}$. The point is that it is apparently easier to control $\vec{h}$ than $\vec{g}$ through heat kernel bounds. On the other hand, if one considers $\mathcal{G}$ instead of its heat kernel analogue, one can directly use Meyer's results; there is, however, little doubt that with more work one could also use Theorem 5.1 or at the opposite an analogous result involving only functionals built with the heat semigroups on functions and on forms, since an analogue of Meyer's result probably holds for the heat semigroup.

THEOREM 5.2 *Let M be a complete Riemannian manifold and $p \in]1, +\infty[$. Assume that $\mathcal{G}$ is bounded on $L^p(M, \mu)$ and that $\vec{h}$ is bounded on $L^q(M; T^*M)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the Riesz transform $\nabla \Delta^{-1/2}$ is bounded on $L^p(M, \mu)$. That is, there exists C_p such that*

$$\| |\nabla f| \|_p \leq C_p \| \Delta^{1/2} f \|_p \quad \forall f \in \mathcal{C}_0^\infty(M) \, .$$

PROOF: One starts with the formula

$$\eta = c \int_0^{+\infty} t \vec{\Delta} \vec{H}_t \sqrt{t \vec{\Delta}} \, \vec{P}_t \eta \, \frac{dt}{t} \, ,$$

which holds in $L^2(M; T^*M)$ by spectral theory, with $\eta = d\Delta^{-1/2} f$, since this form is not harmonic; then one writes

$$\langle \omega, \eta \rangle = c \int_0^{+\infty} \left\langle t\vec{\Delta}\vec{H}_t\omega, \sqrt{t\vec{\Delta}}\,\vec{P}_t\eta \right\rangle \frac{dt}{t}\,.$$

The rest of the proof is similar to the one for Theorem 5.1. □

5.2 Riesz Transform and Heat Kernel on 1-Forms

Assumptions

Let us describe our set of assumptions. The value of the constants C and c may vary from line to line.

Let ρ be the geodesic distance on the complete Riemannian manifold M. Denote by $B(x, r)$ the geodesic ball of center $x \in M$ and radius $r > 0$, and by $V(x, r)$ its Riemannian volume $\mu(B(x, r))$. Let $h_t(x, y)$ be the heat kernel on M, that is, the kernel of the heat semigroup H_t.

(1) DOUBLING PROPERTY. There exists C such that

$$V(x, 2r) \le CV(x, r) \quad \forall x \in M,\ r > 0\,. \tag{5.3}$$

(2) UPPER ESTIMATE FOR THE HEAT KERNEL. There exists C such that

$$h_t(x, x) \le \frac{C}{V(x, \sqrt{t})} \quad \forall x \in M,\ t > 0\,. \tag{5.4}$$

The conjunction of (5.3) and (5.4) is well understood: It is equivalent to a so-called relative Faber-Krahn inequality (see [25]). Under (5.3), (5.4) self-improves into

$$h_t(x, y) \le \frac{C}{V(x, \sqrt{t})} e^{\frac{-c\rho^2(x,y)}{t}} \quad \forall x, y \in M,\ t > 0 \tag{5.4$'$}$$

(see [27]).

(3) GAUSSIAN UPPER ESTIMATES ON THE HEAT KERNEL ON 1-FORMS. There exist C and c such that

$$|\vec{H}_t\omega(x)| \le \frac{C}{V(x, \sqrt{t})} \int_M e^{\frac{-c\rho^2(x,y)}{t}} |\omega(y)| d\mu(y) \tag{5.5}$$

for all $\omega \in L^2(M; T^*M)$, $x \in M$, $t > 0$.

We shall also consider the restriction of condition (5.5) to exact forms:

$$|\vec{H}_t\, df(x)| \le \frac{C}{V(x, \sqrt{t})} \int_M e^{\frac{-c\rho^2(x,y)}{t}} |df(y)| d\mu(y) \tag{5.5$_e$}$$

for all $f \in \mathcal{C}_0^\infty(M)$, $x \in M$, $t > 0$, which is equivalent to

$$|\nabla H_t f(x)| \le \frac{C}{V(x, \sqrt{t})} \int_M e^{\frac{-c\rho^2(x,y)}{t}} |\nabla f(y)| d\mu(y)$$

for all $f \in \mathcal{C}_0^\infty(M)$, $x \in M$, $t > 0$.

(3′) DOMINATION OF THE HEAT SEMIGROUP ON 1-FORMS BY THE HEAT SEMIGROUP ON FUNCTIONS. There exist $C, \theta > 0$ such that

$$|\vec{H}_t\omega(x)| \le C H_{\theta t}|\omega|(x) \quad \forall \omega \in L^2(M; T^*M),\ x \in M,\ t > 0\,. \tag{5.5′}$$

Note that we *do not* require $C = 1$ in (5.5′).

Assumptions (5.3), (5.4), and (5.5′) obviously imply (5.5); on the other hand, one can show that (5.3), (5.4), and (5.5)—in fact, (5.3), (5.4), and (5.5_e)—imply an optimal pointwise upper bound on $\nabla_x h_t(x, y)$ (see Section 5.2 below), which yields

$$h_t(x, y) \ge \frac{c}{V(x, \sqrt{t})}\, e^{\frac{-C\rho^2(x,y)}{t}}\,. \tag{5.4″}$$

Now (5.4″) together with (5.5) implies (5.5′). Finally, (5.3) + (5.4) + (5.5) is equivalent to (5.3) + (5.4) + (5.5′).

We shall also consider the restriction of condition (5.5′) to exact forms:

$$|\vec{H}_t\, df(x)| \le C H_{\theta t}|df|(x) \quad \forall f \in \mathcal{C}_0^\infty(M),\ x \in M,\ t > 0\,, \tag{5.5′$_e$}$$

which is equivalent to

$$|\nabla H_t f(x)| \le C H_{\theta t}|\nabla f|(x) \quad \forall f \in \mathcal{C}_0^\infty(M),\ x \in M,\ t > 0\,, \tag{5.6}$$

that is, a slight modification of condition (1.3) above. Note that the considerations above and Section 5.2 below show that (5.3)+(5.4)+(5.5_e) is equivalent to (5.3)+(5.4) + ($5.5'_e$).

All these assumptions are satisfied if M has nonnegative Ricci curvature. The results below therefore generalize one of the main results of [4]. Conversely, if (5.5′) holds with $C = \theta = 1$, one sees by differentiating and using the Bochner-Lichnerowicz formula that M has nonnegative Ricci curvature. The general version of (5.5′) (say with $C > 1$) carries less information, since no differential version of it is available. On the other hand, it is less rigid, and although little is known at present on heat kernels on 1-forms, one may hope to get estimates like (5.5) in more general circumstances (like perturbations of manifolds with nonnegative Ricci curvature).

Gradient of the Heat Kernel on Functions and Heat Kernel on Forms

In this section we finish the proof of the above assertions that (5.3)+(5.4)+(5.5) is equivalent to (5.3) + (5.4) + (5.5′), and that (5.3) + (5.4) + (5.5_e) is equivalent to (5.3) + (5.4) + ($5.5'_e$). What remains to be proven is that (5.5_e) together with (5.3) and (5.4) implies

$$|\nabla_x h_t(x, y)| \le \frac{C}{\sqrt{t}V(x, \sqrt{t})}\, e^{\frac{-c\rho^2(x,y)}{t}} \quad \forall x, y \in M,\ t > 0\,, \tag{5.7}$$

for some $C, c > 0$. The fact that the gradient upper bound (5.7) together with (5.3) and (5.4) implies the heat kernel lower bound (5.4″) is classical and easy.

Write

$$\nabla_x h_t(x,y) = \int_M \nabla_x h_{t/2}(x,z) h_{t/2}(z,y) d\mu(z) = \nabla_x H_{t/2} f_{y,t}(x)\,, \tag{5.8}$$

where $f(x) = f_{y,t}(x) = h_{t/2}(x,y)$. Assumption ($5.5_e$), together with doubling, yields

$$|\nabla_x H_{t/2} f_{y,t}(x)| \le \frac{C'}{V(x,\sqrt{t})} \int_M e^{\frac{-2c\rho^2(x,z)}{t}} |\nabla_z f_{y,t}(z)| d\mu(z)\,,$$

that is, according to (5.8),

$$|\nabla_x h_t(x,y))| \le \frac{C'}{V(x,\sqrt{t})} \int_M e^{\frac{-2c\rho^2(x,z)}{t}} |\nabla_z h_{t/2}(z,y)| d\mu(z)\,.$$

Thus, it is enough to prove that, for some $\gamma \in]0,c]$, there exist $C, c' > 0$ such that

$$\int_M e^{\frac{-\gamma\rho^2(x,z)}{t}} |\nabla_z h_t(z,y)| d\mu(z) \le \frac{C}{\sqrt{t}} e^{\frac{-c'\rho^2(x,y)}{t}} \quad \forall t > 0,\ x,y \in M.$$

Write $\rho^2(x,y) \le 2(\rho^2(x,z) + \rho^2(z,y))$; therefore

$$\rho^2(x,z) \ge \frac{\rho^2(x,y)}{2} - \rho^2(z,y) \quad \text{and} \quad e^{\frac{-\gamma\rho^2(x,z)}{t}} \le e^{\frac{-\gamma\rho^2(x,y)}{2t}} e^{\frac{\gamma\rho^2(z,y)}{t}}\,.$$

Hence

$$\int_M e^{\frac{-\gamma\rho^2(x,z)}{t}} |\nabla_z h_t(z,y)| d\mu(z) \le e^{\frac{-\gamma\rho^2(x,y)}{2t}} \int_M e^{\frac{\gamma\rho^2(z,y)}{t}} |\nabla_z h_t(z,y)| d\mu(z)\,.$$

But according to [26] (see also [14, lemma 2.4]), it follows from assumptions (5.3) and (5.4) that

$$\int_M e^{\frac{\gamma\rho^2(z,y)}{t}} |\nabla_z h_t(z,y)| d\mu(z) \le \frac{C}{\sqrt{t}}$$

as soon as γ is small enough. The claim is proven.

The Square Function Estimate for the Heat Semigroup on Forms

Recall the definition of the Littlewood-Paley function $\vec{h}$:

$$\vec{h}(\omega)(x) = \left(\int_0^{+\infty} \left| \frac{\partial}{\partial t} \vec{H}_t \omega(x) \right|^2 t\, dt \right)^{\frac{1}{2}}.$$

PROPOSITION 5.3 *Let M be a complete Riemannian manifold satisfying properties* (5.3) *and* (5.5). *Then $\vec{h}$ is bounded on $L^p(M; T^*M)$ for all $p \in]1,+\infty[$.*

The boundedness of $\vec{h}$ on L^2 is a consequence of the self-adjointness of $\vec{H}_t$; it is therefore enough to prove that $\vec{h}$ is of weak type $(1,1)$, and the full result follows by interpolation and duality.

The conclusion of Proposition 5.3, that is,

$$\left\| \left(\int_0^{+\infty} \left| \frac{\partial}{\partial t} \vec{H}_t \omega(\,\cdot\,) \right|^2 t\, dt \right)^{\frac{1}{2}} \right\|_p \leq C_p \|\omega\|_p \quad \forall \omega \in L^p(M; T^*M)\,,$$

is similar to the most classical Littlewood-Paley-Stein inequalities for Markov semigroups (see [41]), except that here $\vec{H}_t$ has values in a vector bundle and no positivity is available. However, one can treat this situation thanks to the methods of [2], where a vector-valued version of [24] is carried out; we sketch a proof here for the sake of completeness.

We shall perform a vector-valued (more precisely a vector-bundle-valued) version of the Calderón-Zygmund decomposition as in [2, sec. 2]. There exists C depending only on the constant in the doubling property (5.3) such that, for every form ω in $L^1(M; T^*M) \cap L^2(M; T^*M)$ and every $\alpha > 0$, one can write

$$\omega = \gamma + \sum_i \beta_i$$

in such a way that the following hold:

(i) $|\gamma(x)| \leq C\alpha \ \forall x \in M$.

(ii) There exists a sequence of balls B_i, $i = 1, 2, \ldots$, such that for all i, $\operatorname{supp}(\beta_i) \subset B_i$ and

$$\int_{B_i} |\beta_i(x)| d\mu(x) \leq C\alpha\mu(B_i)\,.$$

(iii) $\sum_i \mu(B_i) \leq \frac{C}{\alpha} \|\omega\|_1$.

(iv) There exists $N_0 \in \mathbb{N}^*$ such that each $x \in M$ does not appear in more than N_0 balls B_i.

It follows easily from conditions (ii) and (iii) that

$$\max\left\{ \|\gamma\|_1, \sum_i \|\beta_i\|_1 \right\} \leq C\|\omega\|_1\,.$$

One obtains the above decomposition as in [10, theorem 2.2, chap. 3], with two differences: First, one considers the scalar-valued uncentered maximal function

$$\mathcal{M}w(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |\omega(x)| d\mu(x)\,,$$

and second, in our setting, the usual mean zero property of the bad parts β_i does not make sense.

Set

$$\Omega = \{x \in M : \mathcal{M}w(x) > C\alpha\}\,, \quad \gamma = \omega\chi_{\Omega^c}\,,$$

where χ_{Ω^c} is the characteristic function of the complement of Ω, and C is large enough. Property (i) holds because $|\omega(x)| \leq \mathcal{M}w(x)$ for a.e. x.

One then constructs the sequence of balls B_i such that $\Omega = \bigcup_i B_i$ and satisfying (iv) by using a covering lemma of Whitney type; see [10, theorem 1.3, chap. 3]. The B_i's satisfy (iii) because

$$\int_{B_i} |\omega(x)| d\mu(x) \leq C\alpha\mu(B_i)$$

as a consequence of properties of the uncentered maximal function.

One finally sets $\beta_1 = \omega\chi_{B_1}$ and

$$\beta_i = \omega\chi_{B_i \setminus \bigcup_{1\leq j\leq i-1} B_j}\,, \quad i \geq 2,$$

so that (ii) holds and $\omega = \gamma + \sum_i \beta_i$.

We can now show the weak-type (1,1) estimate for $\vec{h}$. This means

$$\mu(\{x \in M : \vec{h}(\omega)(x) > \alpha\}) \leq \frac{C}{\alpha}\|\omega\|_1\,.$$

Since the operator $\vec{h}$ is sublinear,

$$\begin{aligned}\mu(\{x \in M : \vec{h}(\omega)(x) > \alpha\}) &\leq \mu\Big(\Big\{x \in M : \vec{h}(\gamma)(x) > \frac{\alpha}{2}\Big\}\Big)\\ &\quad + \mu\Big(\Big\{x \in M : \vec{h}\Big(\sum_i \beta_i\Big)(x) > \frac{\alpha}{2}\Big\}\Big)\\ &= J_1 + J_2\,.\end{aligned}$$

The estimate on the good part $|J_1| \leq \frac{C}{\alpha}\|\omega\|_1$ is standard. It follows from the L^2-boundedness of $\vec{h}$ and the upper bound on $|\gamma|$ in property (ii). For the bad parts, we first fix an i and then write

$$\beta_i = \vec{H}_{t_i}\beta_i + (I - \vec{H}_{t_i})\beta_i\,,$$

where $t_i = \sqrt{r_i}$, r_i being the radius of the ball B_i.

Using assumption (5.5) on the kernel bound of $\vec{H}_t$, properties (ii) and (iv) of the Calderón-Zygmund decomposition, and the L^2-boundedness of the Hardy-Littlewood maximal function, one can show that

$$\Big\|\sum_i \vec{H}_{t_i}\beta_i\Big\|_2 \leq C\alpha^{1/2}\|\omega\|_1^{1/2}\,.$$

For details, see estimate (2.7) [2, theorem 1], where the estimate was proven in the setting of vector-valued functions. The adaptation to the case of a vector bundle is straightforward. Together with the L^2-boundedness of $\vec{h}$, this gives the weak-type (1,1) estimate

$$\mu\Big(\Big\{x \in M : \vec{h}\Big(\sum_i \vec{H}_{t_i}\beta_i\Big)(x) > \frac{\alpha}{4}\Big\}\Big) \leq \frac{C}{\alpha}\|\omega\|_1\,.$$

Using property (iii) of the Calderón-Zygmund decomposition and the doubling property of M, the weak (1,1) estimate for the last term

$$\mu\Big(\Big\{x \in M : \vec{h}\Big(\sum_i (I - \vec{H}_{t_i})\beta_i\Big)(x) > \frac{\alpha}{4}\Big\}\Big) \leq \frac{C}{\alpha}\|\omega\|_1$$

is straightforward if we can show that, uniformly in i,

$$\int_{2B_i^c} \vec{h}(I - \vec{H}_{t_i})\beta_i(x)d\mu(x) \leq C\|\beta_i\|_1 . \tag{5.9}$$

A similar estimate was proven in [2] (see (4.15) in the proof of theorem 5) with more general functions in place of $\frac{\partial}{\partial t}\vec{H}_t$ in the definition of $\vec{h}$, but since there the assumptions on the underlying space are slightly different, we sketch a proof below.

The fact that (5.9) holds is a consequence of assumption (5.5). First, we observe that assumption (5.5) and the analyticity of $\vec{H}_t$ imply, for $j \in \mathbb{N}^*$,

$$\Big|\frac{\partial^j}{\partial t^j}\vec{H}_t\omega(x)\Big| \leq \frac{C_j}{t^j V(x,\sqrt{t})}\int_M e^{\frac{-c\rho^2(x,y)}{t}}|\omega(y)|d\mu(y) \tag{5.10}$$

for all $\omega \in L^2(M; T^*M)$, $x \in M$, $t > 0$.

We now drop the subscript i. One has

$$\begin{aligned}
\vec{h}[(I - e^{-t\vec{\Delta}})\beta](x) &= \Big(\int_0^{+\infty} \big|\vec{\Delta}(\vec{H}_s - \vec{H}_{s+t})\beta(x)\big|^2 s\,ds\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{k=0}^{\infty}\int_{kt}^{(k+1)t} \big|\vec{\Delta}(\vec{H}_s - \vec{H}_{s+t})\beta(x)\big|^2 s\,ds\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{k=0}^{\infty}\int_{kt}^{(k+1)t} \Big|\int_s^{s+t} \frac{\partial^2}{\partial u^2}\vec{H}_u\beta(x)du\Big|^2 s\,ds\Big)^{\frac{1}{2}} \\
&\leq c\Big(\sum_{k=0}^{\infty}\int_{kt}^{(k+1)t} \Big|\int_s^{s+t}\int_M \frac{1}{u^2 V(x,\sqrt{u})}\, e^{\frac{-c\rho^2(x,y)}{u}}|\beta(y)|d\mu(y)du\Big|^2 s\,ds\Big)^{\frac{1}{2}},
\end{aligned}$$

where the last inequality uses (5.10) for $j = 2$. We note that the right-hand side is now an expression in terms of scalar values. The remainder of the proof is similar to the estimates of [16, secs. 3.3, 3.4], which proved the weak type (1,1) of Littlewood-Paley-Stein square functions (in the case of scalar values) under the assumption of Gaussian heat kernel bounds.

Suppose that instead of (5.5) one only assumes (5.5_e), that is, the same estimate restricted to exact forms. Then the same proof, performed on the Banach space D_p from Section 2.1 (note that this space is preserved by the Calderón-Zygmund

decomposition), yields

$$\left\| \left(\int_0^{+\infty} \left| \frac{\partial}{\partial t} \vec{H}_t \, df(\,\cdot\,) \right|^2 t \, dt \right)^{\frac{1}{2}} \right\|_p \leq C_p \|df\|_p \quad \forall p \in]1, +\infty[\tag{5.11}$$

for all $f \in \mathcal{C}_0^\infty(M)$. Then a simple reformulation of (5.11) yields the following.

PROPOSITION 5.4 *Let M be a complete Riemannian manifold satisfying properties* (5.3) *and* (5.5_{e}). *Then*

$$\left\| \left(\int_0^{+\infty} |\nabla \Delta H_t f(\,\cdot\,)|^2 t \, dt \right)^{\frac{1}{2}} \right\|_p \leq C_p \| |\nabla f| \|_p$$

for all $p \in]1, +\infty[$ and all $f \in \mathcal{C}_0^\infty(M)$.

The above estimate is the analogue for the heat semigroup of certain Littlewood-Paley estimates for the Poisson semigroup, which one can find in [3, pp. 157–158]. It will be used in the proof of Theorem 6.1 below.

The Result

By combining Theorem 5.2, Proposition 3.1, and Proposition 5.3, we obtain the following.

THEOREM 5.5 *Let M be a complete Riemannian manifold satisfying assumptions* (5.3), (5.5), *and* ($5.5'_{\mathrm{e}}$) (*which is the case in particular if* (5.3), (5.4), *and* (5.5) *or if* (5.3), (5.4), *and* (5.5′) *hold*). *Then the Riesz transform $\nabla \Delta^{-1/2}$ is bounded on $L^p(M, \mu)$, $2 < p < +\infty$. That is, for every $p \in]2, +\infty[$, there exists C_p such that*

$$\| |\nabla f| \|_p \leq C_p \|\Delta^{1/2} f\|_p \quad \forall f \in \mathcal{C}_0^\infty(M) \, .$$

To put the contents of Theorem 5.5 in perspective, call R the class of complete Riemannian manifolds with nonnegative Ricci curvature, H the class of manifolds satisfying (5.3)+(5.4)+(5.5) or (5.3)+(5.4)+(5.5_{e}), G the class of manifolds that satisfy the doubling property together with pointwise estimates for the gradient of the heat kernel ((5.7) above), P the class of manifolds satisfying a uniform parabolic Harnack principle (see, e.g., [38]), and F the class of manifolds satisfying a Faber-Krahn inequality (that is, (5.3) and (5.4); see [25]). One has

$$R \subset H \subset G \subset P \subset F$$

(and all inclusions are, or are likely to be, strict).

As we already said, Bakry proved that on all manifolds of the class R, Riesz transforms are L^p-bounded for all $p \in]1, +\infty[$, whereas we proved in [14] that L^p-boundedness for all $1 < p < 2$ extends to the much larger class F and that the L^p-boundedness for $p > 2$ does not. It was observed [18] that there are even manifolds in the class P on which the Riesz transform is unbounded for some $p > 2$. Theorem 5.5 shows that L^p-boundedness for all $p > 2$ holds for the class H. One may still wonder whether this could extend to the class G. Note that it

follows from [31] that there are manifolds that do not belong to the class G but where the Riesz transform is bounded for some $p > 2$.

According to Section 2.1, a corollary of Theorem 5.5 is that, if (5.3), (5.5), and ($5.5'_{\mathrm{e}}$) or (5.3), (5.4), and (5.5) hold, then for every $p \in]1, 2[$, there exists C_p such that

$$\|\Delta^{1/2} f\|_p \leq C_p \||\nabla f|\|_p \quad \forall f \in \mathcal{C}_0^\infty(M) .$$

In the next section we shall see that the latter inequality can be obtained under weaker assumptions.

6 The Reverse Inequalities

6.1 A Positive Result

Recall from Proposition 2.1 that the inequality

$$\|\Delta^{1/2} f\|_p \leq C_p \||\nabla f|\|_p \quad \forall f \in \mathcal{C}_0^\infty(M) \tag{6.1}$$

for $1 < p < 2$ is implied by the boundedness of the Riesz transform for $p > 2$; it is therefore natural to prove the former under slightly weaker assumptions than the latter.

THEOREM 6.1 *Let M be a complete Riemannian manifold satisfying assumptions* (5.3), (5.5_{e}), *and* ($5.5'_{\mathrm{e}}$) (*which is the case in particular if* (5.3), (5.4), *and* (5.5_{e}) *or* (5.3), (5.4), *and* ($5.5'_{\mathrm{e}}$) *hold*). *Then, for every $p \in]1, 2[$, there exists C_p such that* (6.1) *holds.*

REMARK It follows from [14] that, in the case $2 < p < +\infty$, the same inequality holds under assumptions (5.3) and (5.4) only.

PROOF: Let $p \in]1, 2[$ and $q = \frac{p}{p-1}$. We have to estimate

$$\|\Delta^{1/2} f\|_p = \sup_{\|g\|_q \leq 1} \langle g, \Delta^{1/2} f \rangle .$$

Write, by spectral theory,

$$f = c \int_0^{+\infty} \Delta^{3/2} P_t H_t f t \, dt ;$$

thus

$$\begin{aligned}
\langle g, \Delta^{1/2} f \rangle &= c \int_0^{+\infty} \big\langle P_t g, \Delta^2 H_t f \big\rangle t \, dt \\
&= c \int_0^{+\infty} \int_M \nabla P_t g(x) \cdot \nabla \Delta H_t f(x) d\mu(x) t \, dt \\
&\leq c \int_0^{+\infty} \int_M |\nabla P_t g(x)| |\nabla \Delta H_t f(x)| t \, dt \, d\mu(x)
\end{aligned}$$

$$\leq c \int_M \left(\int_0^{+\infty} |\nabla P_t g(x)|^2 t \, dt \right)^{\frac{1}{2}} \left(\int_0^{+\infty} |\nabla \Delta H_t f(x)|^2 t \, dt \right)^{\frac{1}{2}} d\mu(x)$$

$$\leq c \|\mathcal{G}(g)\|_q \left\| \left(\int_0^{+\infty} |\nabla \Delta H_t f(\cdot)|^2 t \, dt \right)^{\frac{1}{2}} \right\|_p .$$

Now, an easy modification of (ii) in Proposition 3.1 shows that assumption ($5.5'_{\mathrm{e}}$) yields

$$\sup_{\|g\|_q \leq 1} \|\mathcal{G}(g)\|_q < +\infty$$

and, according to Proposition 5.4, (5.3) and (5.5_{e}) yield

$$\left\| \left(\int_0^{+\infty} |\nabla \Delta H_t f(\cdot)|^2 t \, dt \right)^{\frac{1}{2}} \right\|_p \leq C \| |\nabla f| \|_p .$$

Finally,

$$\|\Delta^{1/2} f\|_p \leq C' \| |\nabla f| \|_p .$$

□

The same remark as the one before Theorem 5.2 applies; with a heat semigroup analogue of the functional $\mathcal{G}$, or with a Poisson semigroup analogue of Proposition 5.4, one could write down a more symmetric proof.

6.2 A Counterexample

The following example shows that assumption (5.3) would not suffice in Theorem 6.1.

PROPOSITION 6.2 *For every $p_0 \in]1, 2[$, there exists a complete noncompact Riemannian manifold with bounded geometry and polynomial volume growth such that the inequality*

$$\|\Delta^{1/2} f\|_p \leq C_p \| |\nabla f| \|_p \quad \forall f \in \mathcal{C}_0^\infty(M) \tag{6.2}$$

is false for every $p \in]1, p_0[$.

A similar example is contained in [17, sec. 5], but it does not satisfy polynomial growth or the doubling property. Also, the above counterexample is stronger than the counterexample in [14, sec. 5], since here we violate a weaker inequality.

PROOF: Let M be a Vicsek manifold as considered in [6, theorem 6.3]. It has polynomial growth of exponent $D > 1$ for some (arbitrarily large) D,

$$V(x, r) \simeq r^D , \quad r \geq 1,$$

and it satisfies

$$\sup_{x \in M} p_t(x, x) \simeq t^{-\frac{D}{D+1}} , \quad t \geq 1.$$

Let $D' = \frac{2D}{D+1}$. According to [8] (see also [13, sec. III]), the following Nash inequality holds on M:

$$\|f\|_2^{1+(2/D')} \leq C\|f\|_1^{2/D'}\||\nabla f|\|_2 \quad \forall f \in \mathcal{C}_0^\infty(M) \text{ such that } \frac{\|f\|_2}{\|f\|_1} \leq 1\,.$$

That is,

$$\|f\|_2^{1+(2/D')} \leq C\|f\|_1^{2/D'}\|\Delta^{1/2} f\|_2 \quad \forall f \in \mathcal{C}_0^\infty(M) \text{ such that } \frac{\|f\|_2}{\|f\|_1} \leq 1\,.$$

More generally, the proof of [11, théorème 2] shows that, for $p \in]1, +\infty[$,

$$\|f\|_p^{1+p/(p-1)D'} \leq C_p\|f\|_1^{p/(p-1)D'}\|\Delta^{1/2} f\|_p$$

for all $f \in \mathcal{C}_0^\infty(M)$ such that $\|f\|_p/\|f\|_1 \leq 1$.

Assume now that (6.2) holds. It follows that

$$\|f\|_p^{1+p/(p-1)D'} \leq C'_p\|f\|_1^{p/(p-1)D'}\||\nabla f|\|_p \tag{6.3}$$

for all $f \in \mathcal{C}_0^\infty(M)$ such that $\|f\|_p/\|f\|_1 \leq 1$.

Consider a discretization Γ of the manifold M (see [12, 20, 29]); here Γ can be identified with the Vicsek graph out of which M was built (see [6]). From (6.3) one obtains

$$\|\varphi\|_p^{1+p/(p-1)D'} \leq C'_p\|\varphi\|_1^{p/(p-1)D'}\|\nabla_\Gamma\varphi\|_p \quad \forall\varphi \in c_0(\Gamma)\,, \tag{6.4}$$

where ∇_Γ is the discrete gradient defined by

$$\nabla_\Gamma\varphi(x) = \sum_y |\varphi(x) - \varphi(y)|\,,$$

where the summation ranges over all neighbors y of x in the graph Γ.

Now $\|\nabla_\Gamma\chi_\Omega\|_p \simeq |\partial\Omega|^{1/p}$ for $\Omega \subset \Gamma$, and if one applies (6.4) to χ_Ω, one obtains

$$|\Omega|^{1-p/D'} \leq C_p|\partial\Omega|\,. \tag{6.5}$$

Here $\partial\Omega$ is the set of vertices in Ω that have a neighbor outside Ω, and $|A|$ is the cardinality of a subset of vertices A in Γ. But a basic property of the Vicsek graph is that it contains sets Ω_n such that $|\Omega_n|$ tends to infinity whereas $|\partial\Omega_n|$ is constant. As a consequence, for $p < D'$, (6.5) cannot be true, that is, (6.2) is false. Finally, D may be chosen arbitrarily large, and thus D' arbitrarily close to 2, which proves the claim. □

Note Added in Proof

Several developments have occurred since the present paper was finished. In a preprint called "Riesz transform on manifolds and heat kernel regularity" by P. Auscher, T. Coulhon, X-T. Duong, and S. Hofmann, among other results, Conjecture 1.1 has been proven in the case when $p > 2$ for manifolds with the doubling property and it has been shown that, for such manifolds, condition (1.5) is implied

by Gaussian upper estimates of the heat kernel and its gradient. As a consequence, the Riesz transform is bounded on L^p for all p, $1 < p < +\infty$, on manifolds of the class G considered at the end of Section 5.2. It also follows from a recent result of N. Dungey that cocompact covering manifolds with polynomial growth belong to the class G and therefore satisfy (1.5), which answers the question we were asking at the end of the introduction. Finally, a new proof of (a slightly stronger form of) Theorem 5.5, based on the finite-speed propagation of solutions of the wave equation, was given by A. Sikora in the latest version of [39].

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THIERRY COULHON
Université de Cergy-Pontoise
Département de Mathématiques
2, rue Adolphe Chauvin
95302 Pontoise
FRANCE
E-mail: `Thierry.Coulhon@math.u-cergy.fr`

XUAN THINH DUONG
Macquarie University
Department of Mathematics
North Ryde, New South Wales 2109
AUSTRALIA
E-mail: `duong@ics.mq.edu.au`

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