Geometric interpretations of L^p-Poincaré inequalities on graphs with polynomial volume growth

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Abstract

On graphs with polynomial volume growth of exponent Q, we characterize L^p-Poincaré inequalities in the range $Q-1 < p < +\infty$ in terms of capacity estimates, depending on the position of p with respect to Q.

Contents

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1 Introduction

Let (X, d, μ) be a metric measured space. Denote by $B(x, r)$ the ball of radius $x \in X$ and radius $r > 0$, and, if f is a measurable function on X, denote by $f_{B(x,r)}$ the mean of f on $B(x, r)$. One says that X satisfies the L^p-Poincaré inequality (P_p), for some $p \in [1, +\infty[$, if there exists $C > 0$ such that

$$\int_{B(x,r)} |f - f_{B(x,r)}|^p \, d\mu \leq C r^p \int_{B(x,r)} |\nabla f|^p \, d\mu$$

whenever f is a sufficiently regular function on X, $x \in X$, and $r > 0$. Here $|\nabla f|$ denotes the upper gradient of f; this notion was recently designed to work in all metric mesured spaces (see for instance [22]), but in more concrete situations one may think of the usual gradient. General references on Poincaré inequalities in the context of metric spaces include [22] and [23].

Poincaré inequalities have recently played an important role in several aspects of analysis on non-compact manifolds and infinite graphs. Together with the doubling volume property, (P_1) ensures a good connection between volume lower bounds and isoperimetric inequalities ([12]). The conjunction of (P_2) and of the doubling volume property is equivalent to the parabolic Harnack inequality, which is itself equivalent to Gaussian upper and lower pointwise estimates of the heat kernel on manifolds ([34], [19], [36]), resp. of random walks on graphs ([14]); (P_2) was also used in the study of harmonic functions ([6]). If $1 < p < +\infty$, (P_p) and the doubling volume property imply elliptic Harnack inequalities for p-Laplace operators (see for instance [13] and the references therein). Finally, if X is a manifold with polynomial growth of exponent Q, (P_Q) plays an important role in the theory of quasiconformal and quasiregular mappings ([25], [13]). Note at once that (P_p) is stronger and stronger as p decreases (see Section 2.4 below).

It is therefore challenging to obtain a good geometric understanding of these inequalities.

Following ideas introduced by Maz'ya, one can show that (P_p) is equivalent to a lower bound of the p-capacity between two sets sitting in a ball, in terms of the volume of the sets and the radius of the ball (see Section 2.5 below). Our main objective in this paper is to show that if X has polynomial growth of exponent Q and p is larger than $Q - 1$, one can give more tractable characterizations of (P_p), which do not involve the volume of $|E|$ and $|F|$ any more, but instead some lower dimensional geometric features of E and F, like their diameter and the distance between them.

The good relationship between the metric and the gradient on a space X is shown by the fact that the distance

$$\delta_\infty(x, y) = \sup\{f(x) - f(y); \| |\nabla f| \|_\infty \leq 1\}$$

is equivalent to the original metric d. If one replaces the L^∞ norm by an L^p norm in the above definition, one gets a family of distances δ_p on X, for $p \in [1, +\infty]$. Instead of considering $\delta_p(x, y)$ for x, y points in X, one can more generally consider $\delta_p(E, F)$ for E, F subsets of X. In fact, $\delta_p(E, F) = \frac{1}{\left(\mathrm{Cap}_p(E,F)\right)^{1/p}}$, where Cap_p is the more familiar p-capacity of a pair of sets.

Suppose that X has polynomial volume growth of exponent Q. For $p > Q$, the connection between (P_p) and the fact that $\delta_p(x, y)$ is uniformly comparable to $d(x, y)^{1-\frac{Q}{p}}$ has been made in [7]. The case $p = Q$ has been treated in [25]: it follows from the main result here (in a slightly different setting) that (P_Q) is equivalent to a lower estimate of $\delta_Q(E, F)$ by

$$\left(\log \frac{d(E, F)}{\min(\operatorname{diam}(E), \operatorname{diam}(F))}\right)^{\frac{Q-1}{Q}},$$

for E, F disjoint and connected subsets of X.

Our aim is to put these two theorems into a common framework, and to make more systematic the relationship between estimates on δ_p and Poincaré inequalities (P_p). The picture strongly depends on the position of p with respect to the volume growth exponent Q. For $Q < p < +\infty$, (P_p) can be characterized in terms of upper estimates on δ_p on pairs of points (Section 3). To characterize (P_Q), one has to consider pairs of connected sets E, F and one-dimensional features associated with them, namely their diameters and their relative distance (Section 4). For $Q - 1 < p < Q$, (P_p) also admits a characterization, in terms of an estimate involving pairs of sets E, F, a ball containing them, and their diameters (Section 5.1). Below $Q - 1$, (P_p) still implies sharp upper estimates on δ_p (or lower estimates on Cap_p) of pairs of sets in terms of their diameters, but the latter do not imply (P_p) in turn any more (Sections 5.2, 5.3). Note that similar limitations on the exponents appear in [33] and [17].

We have chosen to present our results on graphs, because some phenomena appear more easily in a discrete setting. However, these phenomena may have repercussions in the continuous world, through discretisation procedures. We leave this direction for future work.

Section 3 follows [7], with some improvements. Section 4 is a discrete version of [25], that incorporates some ideas in [7] and uses some results of [2] and [22]. The results contained in the rest of the paper as well as the overall perspective are new. In Section 6, we apply our techniques to the following question: if one glues along a set two graphs with polynomial growth of exponent Q, satisfying (P_p), how big must be this set so that the resulting graph still satisfies (P_p)? Again, the answer depends on the position of p with respect to Q.

Most of our arguments can be adapted to the case of so-called β-Poincaré inequalities, which, for $p = 2$, read

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 \, d\mu \leq C r^\beta \int_{B(x,r)} |\nabla f|^2 \, d\mu,$$

with $\beta \geq 2$. Such inequalities appear naturally in the context of fractals. As a matter of fact, this adaptation has recently been started in the preprint [3], see Lemma 2.3 and Section 2.2. We will not pursue this here.

For more characterizations of Poincaré inequalities, see [29].

2 Preliminaries

2.1 Notation

Let Γ be an infinite graph. Assume that Γ is locally uniformly finite, or has bounded degree, i.e. there exists $N \in \mathbb{N}^*$ such that every $x \in \Gamma$ has at most N neighbours. Write $x \sim y$ if $x, y \in \Gamma$ are neighbours. A path of length n between x and y in Γ is a sequence x_i, $i = 0, .., n$ such that $x_0 = x$, $x_n = y$, $x_i \sim x_{i+1}$, $i = 0, .., n-1$. We shall suppose that Γ is connected, i.e. any two points in Γ can be joined by a path, and we shall say that a subset of vertices Ω of Γ is connected if any two vertices $x, y \in \Omega$ can be joined by a path in Γ all of whose vertices are in Ω. Let d be the natural metric on Γ: $d(x, y)$ is the minimal length of a path joining x and y.

To be consistent with the theory of random walks of graphs, where Poincaré inequalities are important, see [11], [14], [9], [1], we shall endow vertices (i.e. elements of Γ), and edges, i.e. pairs of neighbours in Γ, with weights. They will however play no essential role in what follows. In particular, in the examples we shall consider, we will always equip our graphs with the weight constantly equal to one.

For every pair of neighbours $x, y \in \Gamma$, let $\mu_{xy} = \mu_{yx} > 0$. Define

$$\mu(x) = \sum_{y \sim x} \mu_{xy},$$

and

$$p(x, y) = \frac{\mu_{xy}}{\mu(x)}.$$

We shall make the following standing assumption on the weights:

$$p(x, y) \geq p_0 > 0 \text{ for all } \; x, y \in \Gamma, \; x \sim y. \tag{p_0}$$

The space of all functions on Γ will be denoted by $\mathbb{R}^\Gamma$, the space of finitely supported functions by $c_0(\Gamma)$.

For Ω a subset of Γ, define its boundary by

$$\partial\Omega = \{x \in \Omega; \; \exists y \in \Gamma \setminus \Omega, \, y \sim x\}$$

and its interior by

$$\Omega^o = \Omega \setminus \partial\Omega.$$

Denote by $c(\Omega)$ the space of functions on Γ that are supported in Ω^o.

We shall take the ℓ^p norms of functions in $\mathbb{R}^\Gamma$ with respect to the measure μ:

$$\|f\|_p = \left(\sum_{x \in \Gamma} |f(x)|^p \mu(x) \right)^{1/p}, \; 1 \leq p < +\infty, \; \|f\|_\infty = \sup_{x \in \Gamma} |f(x)|.$$

For Ω a subset of Γ, $\|f\|_{p,\Omega}$ will denote $\|f 1_{\Omega^o}\|_p$.

2.2 Capacities, p-distances

For f a function on Γ, define the length of its gradient at $x \in \Gamma$ by

$$|\nabla f|(x) = \sum_{y\in\Gamma,\ y\sim x} |f(x) - f(y)|.$$

Note that, in a context of analysis of the random walk on (Γ, μ), on would rather consider the L^2 quantity $\left(\sum_{y\in\Gamma,\ y\sim x} |f(x) - f(y)|^2 p(x,y)\right)^{1/2}$. Since Γ has bounded degree and given assumption (p_0), this would change nothing to what follows but constants, and the above simpler expression for the gradient is sufficient for our present purpose.

Definition 2.1 *For E, F two disjoint subsets of Γ and $p \in [1, +\infty[$, let*

$$\mathrm{Cap}_p(E, F) := \inf\{|||\nabla f|||_p^p; f \in \mathbb{R}^\Gamma, f\ |_E \geq 1, f\ |_F \leq 0\}.$$

Note that

$$\mathrm{Cap}_p(E, F) = \inf\{|||\nabla f|||_p^p; f \in \mathbb{R}^\Gamma, 0 \leq f \leq 1, f\ |_E = 1, f\ |_F = 0\}.$$

Indeed, since one takes the infimum on a smaller set, the right hand side is larger or equal. The opposite inequality holds because, to a function f such that $f\ |_E \geq 1$ and $f\ |_F \leq 0$, one can always associate $\tilde{f} = \max(\min(f, 1), 0)$, and $|||\nabla \tilde{f}|||_p^p \leq |||\nabla f|||_p^p$.

More generally, if G is a connected subset of Γ, and if E, F are two disjoint subsets of G^0, we shall consider

$$\mathrm{Cap}_p(E, F; G) := \inf\{|||\nabla f|||_{p,G}^p; f \in \mathbb{R}^\Gamma, f\ |_E \geq 1, f\ |_F \leq 0\}.$$

Note that if one changes E, F or G to a smaller set, $\mathrm{Cap}_p(E, F; G)$ decreases.

For more information on p-capacities, see for instance [24].

We shall need (in Proposition 4.8 below) the following canonical way to build test functions for the capacity. A reader familiar with the notion of modulus of curve families should consider this property as a substitute for the fact that the modulus dominates the capacity (see [25], prop. 2.17).

Let $P = \{x_0, ..., x_k\}$ with $x_{i-1} \sim x_i$, $i = 1, ..., k$, be a path in Γ. Say that P joins E to F in G if $x_0 \in E$, $x_k \in F$, and $x_i \in G$, $i = 1, ..., k$. Also, for $f \in \mathbb{R}^\Gamma$, denote by $\sum_P |\nabla f|_P$ the integral of the gradient of f along P, i.e. $\sum_{i=1}^k |f(x_i) - f(x_{i-1})|$.

Proposition 2.2 *Let G be a finite connected subset of Γ and $E, F \subset G^0$ be disjoint. Let $f \in \mathbb{R}^\Gamma$ be such that $\sum_P |\nabla f|_P \geq 1$ for each path P that joins E to F in G. Then, for any $p \in [1, +\infty[$,*

$$|||\nabla f|||_{p,G}^p \geq \mathrm{Cap}_p(E, F; G).$$

Proof: Define $u \in \mathbb{R}^\Gamma$ by

$$u(x) = \min \left\{ \sum_{P_x} |\nabla f|_{P_x} \right\},$$

if $x \in G$, where the infimum is taken over all paths P_x that join x to F in G, and $u(x) = 0$ if $x \notin G$. Then u is a test function for $\mathrm{Cap}_p(E, F; G)$. Indeed, $u\,|_F = 0$ is trivial and $u\,|_E \geq 1$ follows from the assumption. Thus

$$\| |\nabla u| \|_{p,G}^p \geq \mathrm{Cap}_p(E, F; G).$$

Let now $x, y \in G$ be neighbours. Let $P = \{x_0, ..., x_k\}$ be a path joining $x_0 = x$ to F in G, such that $u(x) = \sum_P |\nabla f|_P$. Then $P' = \{y, x_0, ..., x_k\}$ joins y to F in G, and

$$\sum_{P'} |\nabla f|_{P'} = |f(y) - f(x)| + u(x),$$

thus

$$u(y) \leq u(x) + |f(y) - f(x)|.$$

Exchanging the roles of x and y, one sees that

$$|u(x) - u(y)| \leq |f(x) - f(y)|.$$

If $x \in G^o$, all its neighbours y belong to G, and one gets

$$|\nabla u|(x) \leq |\nabla f|(x),$$

which proves the claim.

Definition 2.3 *For E, F two disjoint subsets of Γ and $p \in [1, +\infty]$,*

$$\delta_p(E, F) = \sup\{\inf_{x \in E} f(x) - \sup_{y \in F} f(y); f \in \mathbb{R}^\Gamma, \| |\nabla f| \|_p \leq 1\}.$$

Note that $\delta_p(E, F) > 0$ as soon as E or F is finite (take for f a suitable multiple of the characteristic function of E or F). In the case $E = \{x\}$, $F = \{y\}$, the above quantity has been considered in [7], Section 6, see also [33], Section 4. We shall denote

$$\delta_p(x, y) = \delta_p(\{x\}, \{y\}) = \sup\{f(x) - f(y); f \in \mathbb{R}^\Gamma, \| |\nabla f| \|_p \leq 1\}.$$

For every $p \in [1, +\infty]$, δ_p is a distance. One easily checks that under our assumptions δ_∞ is uniformly comparable to d and that δ_1 is always bounded, if there exists $c > 0$ such that $\mu_{xy} \geq c$ for all neighbours x, y.

The following property is obvious.

Proposition 2.4 *For E, F two disjoint subsets of Γ and $p \in [1, +\infty[$,*

$$\mathrm{Cap}_p(E, F) = \frac{1}{\delta_p^p(E, F)}.$$

The quantity $\mathrm{Cap}_p(\{x\}, \{y\}) = \frac{1}{\delta_p^p(\{x\}, \{y\})}$ is also called the Hölder invariant (see [33]).

2.3 Volume growth

Denote by $B(x,r)=\{y\in\Gamma;\, d(x,y)\le r\}$ the closed ball of center $x\in\Gamma$ and radius $r>0$, and define its volume

$$V(x,r)=\mu(B(x,r))=\sum_{y\in B(x,r)}\mu(y).$$

One says that Γ satisfies the doubling property (or is a space of homogeneous type) if there exists C such that

$$(D)\qquad V(x,2r)\le C\,V(x,r),\ \forall\, x\in\Gamma,\ r>0.$$

One says that Γ has polynomial growth of exponent Q (or is Ahlfors-David Q-regular) if there exists C such that

$$(V_Q)\qquad C^{-1}r^Q\le V(x,r)\le Cr^Q,\ \forall\, x\in\Gamma,\ r>0.$$

We shall distinguish between the volume lower bound of exponent Q, i.e.

$$(LV_Q)\qquad C^{-1}r^Q\le V(x,r),\ \forall\, x\in\Gamma,\ r>0,$$

and the volume upper bound of exponent Q, i.e.

$$(UV_Q)\qquad V(x,r)\le Cr^Q,\ \forall\, x\in\Gamma,\ r>0.$$

Since Γ is infinite and connected, one has at least (LV_1), say if there exists $c>0$ such that $\mu_{xy}\ge c$ for all neighbours x,y. Therefore we may consider only the range $Q\ge 1$.

2.4 Poincaré inequalities

For simplicity, we shall denote in what follows by $\sum_\Omega f$ the sum of $f\in\mathbb{R}^\Gamma$ on $\Omega\subset\Gamma$ with respect to μ:

$$\sum_\Omega f=\sum_{y\in\Omega}f(y)\mu(y).$$

Depending on the context, we shall denote the mean

$$\frac{1}{V(x,r)}\sum_{B(x,r)}f$$

of a function $f\in\mathbb{R}^\Gamma$ on a ball $B=B(x,r)$, $x\in\Gamma$, $r>0$, by

$$f_r(x),\ f_{B(x,r)},\ f_B,\ \text{or}\ \sum_{B(x,r)}{}' f.$$

For $1\le p<+\infty$, one says that Γ satisfies the Poincaré inequality at the level p if there exists C,C' such that

$$(P_p)\qquad \left(\sum_{B(x,r)}|f-f_r(x)|^p\right)^{1/p}\le Cr\left(\sum_{B(x,C'r)}|\nabla f|^p\right)^{1/p},\ \forall x\in\Gamma,\ r>0,\ f\in\mathbb{R}^\Gamma.$$

Note that the obvious modification one may write for $p = +\infty$ always holds under assumption (p_0).

If (D) holds and $1 \leq p < +\infty$, an equivalent form of this inequality is the apparently weaker

$$\sideset{}{'}\sum_{B(x,r)} |f - f_r(x)| \leq Cr \left(\sideset{}{'}\sum_{B(x,C'r)} |\nabla f|^p \right)^{1/p}, \ \forall x \in \Gamma,\ r > 0,\ f \in \mathbb{R}^\Gamma. \tag{2.1}$$

See [21] for the equivalence.

If is clear on the latter form that (P_p) implies (P_q) if $p < q$. The converse is false. See [22], and also [7]. On the other hand, the interval of p's such that (P_p) holds is open (see [28]), so that (P_p) for $p > 1$ always implies $(P_{p-\varepsilon})$ for some $\varepsilon > 0$. Let us note in passing that the latter property may be very useful, since it may be substantially easier to prove certain facts under $(P_{p-\varepsilon})$ rather than (P_p), see for instance [1], Proposition 6.3.

Note that in both forms of (P_p), the constant C' can be taken equal to one, at the expense of enlarging C. This result was originally proved by Jerison in [26] for some subelliptic differential operators, and a simple proof in a general setting was given in [22].

The following characterization of Poincaré inequalities in terms of maximal functions is proved in [22], [25], see also [20]. Define $M_R f(x) := \sup_{r<R} f_r(x)$.

Lemma 2.5 *Suppose that Γ satisfies (D). Then, for any $p \in [1, +\infty[$, (P_p) holds if and only if there exists C such that*

$$|f(x) - f(y)| \leq C\, d(x,y) \left((M_{Cd(x,y)} |\nabla f|^p)(x) + (M_{Cd(x,y)} |\nabla f|^p)(y) \right)^{1/p}, \tag{2.2}$$

$\forall\, f \in \mathbb{R}^\Gamma$, $x, y \in \Gamma$.

Note that the proof that the above condition is sufficient to get (2.1) goes by mere integration, see [22].

2.5 Maz'ya type characterization of Poincaré inequalities

Maz'ya introduced the idea that a Poincaré inequality can be described in terms of capacity (c.f. [31], Section 4.7.5). The following result is a quantitative version of this fact.

Proposition 2.6 *For any $p \in [1, +\infty[$, (P_p) is equivalent to*

$$\mathrm{Cap}_p(E, F; B(x, Cr)) \geq c \frac{\min\{|E|, |F|\}}{r^p}, \tag{2.3}$$

for some $C \geq 1$, $c > 0$, all $x \in \Gamma$, all $r > 0$, and every pair E, F of disjoint subsets of $B(x,r)$.

To see the necessity of (2.3), simply apply (P_p) to a test function for the capacity and notice that either $|f_{B(x,r)}| \geq 1/2$ or $|1 - f_{B(x,r)}| \geq 1/2$. The sufficiency follows from truncations techniques as in [22], Section 12, or [2]. See also the remark after Proposition 5.1 below.

Note that the above characterization is independent of the volume growth exponent, if any. By contrast, the form of the more precise characterizations we are going to give below will depend on the position of the exponent p with respect to the volume growth exponent Q. For instance, the presence of the ball $B(x, Cr)$ in the estimate can be got rid only if $p \geq Q$.

2.6 Sobolev inequalities

Recall that $c_0(\Gamma)$ denotes the space of finitely supported functions on Γ, and that $c(\Omega)$ denotes the space of functions on Γ supported in the interior of the subset Ω.

In all this section, we assume that Γ satisfies (D) and (LV_Q). Then, for fixed $p \in [1, +\infty[$, (P_p) implies the inequality

$$(S_Q^p) \qquad \|f\|_p \leq C|\Omega|^{1/Q}\||\nabla f|\|_p, \ \forall\, \Omega \subset \Gamma, \ f \in c(\Omega).$$

One can see that (S_Q^1) is equivalent to the isoperimetric inequality

$$|\Omega|^{\frac{Q-1}{Q}} \leq C|\partial\Omega|, \ \forall\, \Omega \subset \Gamma,$$

and that (S_Q^∞) is nothing but (LV_Q). Moreover, (S_Q^p) implies (S_Q^q) for $1 \leq p \leq q \leq +\infty$. For $1 \leq p < Q$, (S_Q^p) is equivalent to the more familiar Sobolev inequality

$$\|f\|_{\frac{pQ}{p-Q}} \leq C\||\nabla f|\|_p, \forall f \in c_0(\Gamma),$$

for $p = Q$, to the Trudinger-Moser inequality (here one assumes $Q > 1$)

$$\sum_{x\in\Omega} \exp\left[\left(\frac{c|f(x)|}{\||\nabla f|\|_Q}\right)^{\frac{Q}{Q-1}}\right] \leq C|\Omega|, \ \forall\, \Omega \subset \Gamma, \ f \in c(\Omega),$$

and for $p > Q$, to the Gagliardo-Nirenberg inequality

$$\|f\|_\infty \leq C_p\|f\|_p^{1-\frac{Q}{p}}\||\nabla f|\|_p^{\frac{Q}{p}}, \ \forall f \in c_0(\Gamma).$$

For all this, see [12], [7], [2], [8], [21], [22], [4], and [10].

It is clear that, in the presence of (D) and (LV_Q), (P_p) is strictly stronger than (S_Q^p). Indeed, for $1 \leq p < Q$, (P_p) implies the global Sobolev inequality

$$\|f - c(f)\|_{\frac{pQ}{p-Q}} \leq C\||\nabla f|\|_p, \forall f \in \mathbb{R}^\Gamma,$$

where $c(f)$ is some number; note that f is not assumed to be finitely supported (see [35], and also [7], p.95). A simple example of graph that satisfies (S_Q^p) but fails to satisfy the global Sobolev inequality is obtained by glueing two copies of $\mathbb{Z}^Q$, $Q \geq 2$, along a single edge. This example also shows that (P_Q) is strictly stronger than (S_Q^Q). For $p > Q$, this example satisfies (P_p), but by adding more copies of $\mathbb{Z}^Q$ one obtains an example that satisfies (S_Q^p) but fails to satisfy (P_p): take the graph $\mathbb{Z}$, and attach to each vertex a copy of $\mathbb{Z}^Q$, $Q \geq 2$. Then (S_Q^p) holds on this graph Γ for all $p \in [1, +\infty]$. Indeed, for $f \in \mathbb{R}^\Gamma$, write $f = \sum_k f_k$, where f_k is the restriction of f to the k-th copy of $\mathbb{Z}^Q$. Let Ω be the support of f and Ω_k the support of f_k. Then one has $\|f\|_p \leq \sum_k \|f_k\|_p$, $\Omega = \cup_k \Omega_k$ and $|\nabla f|(x) \geq |\nabla f_k|(x)$ for all $x \in \Gamma$ and $k \in \mathbb{Z}$, so that one deduces (S_Q^p) on Γ from (S_Q^p) on $\mathbb{Z}^Q$ by writing

$$\|f\|_p \leq \sum_k \|f_k\|_p \leq C\sum_k |\Omega_k|^{1/Q}\||\nabla f_k|\|_p \leq C\left(\sum_k |\Omega_k|\right)^{1/Q}\||\nabla f|\|_p.$$

Now (P_p) fails on Γ as one easily checks using the Lipschitz estimate below (take a function on the line, and make it constant on the copies on $\mathbb{Z}^Q$).

For $p > Q$, (P_p) implies the Lipschitz embedding

$$|f(x) - f(y)| \leq C\, d(x,y)^{1-\frac{Q}{p}} \||\nabla f|\|_p,\ \forall f \in \mathbb{R}^\Gamma,\ \forall\, x, y \in \Gamma,$$

and for $p = Q$,

$$|f(x) - f(y)| \leq C(\log d(x,y) + 1)^{\frac{Q-1}{Q}} \||\nabla f|\|_Q,\ \forall f \in \mathbb{R}^\Gamma,\ \forall x, y \in \Gamma,\ x \neq y, \tag{2.4}$$

(see [7], and [2] to obtain the sharp exponent in the second inequality). One may ask whether these properties are strong enough to imply (P_p) in turn. We shall see below that it is the case only if $p > Q$.

3 $Q < p < +\infty$

The main result in this section is the following. This is an improvement upon [7], where the necessity of condition (3.1) was proved, but not its sufficiency.

Theorem 3.1 *Suppose that Γ has polynomial growth of exponent Q. Then, for $Q < p < +\infty$, (P_p) is equivalent to*

$$\delta_p(x,y) \leq C d(x,y)^{1-\frac{Q}{p}},\ \forall\, x, y \in \Gamma. \tag{3.1}$$

It was observed in [7], Section 6, that the converse estimate holds as soon as the volume growth is at most polynomial of exponent Q; compare also with [33], Prop. 4.7. We recall the proof for the sake of completeness.

Proposition 3.2 *The volume upper bound (UV_Q) implies*

$$\delta_p(x,y) \geq c\, d(x,y)^{1-\frac{Q}{p}},\ \forall\, x, y \in \Gamma, p \in [1, +\infty].$$

Proof: Fix x, y in Γ, and consider $f \in \mathbb{R}^\Gamma$ defined by

$$f(z) = (d(x,y) - d(x,z))_+ .$$

Since $|\nabla f| \leq N$ on $B(x, d(x,y))$ (recall that Γ is locally uniformly finite), and $|\nabla f| = 0$ elsewhere, one has $\||\nabla f|\|_p \leq N\, V(x, d(x,y))^{1/p}$.

Now $|f(x) - f(y)| = d(x,y)$, therefore

$$\delta_p(x,y) \geq \frac{d(x,y)}{N\, V(x, d(x,y))^{1/p}}.$$

The claim follows from this inequality and (UV_Q).
□

A consequence of Proposition 3.2 is that the diameter of the metric space (Γ, δ_p) is infinite if $p > Q$ and (UV_Q) holds. Also, it follows from Theorem 3.1 and Proposition 3.2 that

$$(V_Q) + (P_p) \Rightarrow \delta_p(x,y) \simeq d(x,y)^{1-\frac{Q}{p}}.$$

See [15], [33], [16] for applications of such an estimate in a continuous setting.

Let us now prove Theorem 3.1; first for the necessity of (3.1).

Proposition 3.3 *If $p > Q$, (P_p), (LV_Q) and (D) imply*

$$\delta_p(x,y) \le C d(x,y)^{1-\frac{Q}{p}}, \ \forall\, x,y \in \Gamma.$$

Proofs: The proof in [7] goes like this. One notices first that (P_p) and (D) imply

$$\|f - f_r\|_p \le Cr\||\nabla f|\|_p, \ \forall\, f \in c_0(\Gamma), \ \forall r > 0. \tag{PP_p}$$

Then (PP_p) and (LV_Q) imply the Gagliardo-Nirenberg inequality

$$\|f\|_\infty \le C\|f\|_p^{1-\frac{Q}{p}}\||\nabla f|\|_p^{\frac{Q}{p}}, \ \forall f \in c_0(\Gamma). \tag{S_Q^p}$$

Finally, replacing f in (S_Q^p) by $(f - f_r(x))\varphi$, where φ is a suitable cut-off function, yields the Lipschitz embedding

$$|f(x) - f(y)| \le C\, d(x,y)^{1-\frac{Q}{p}}\||\nabla f|\|_p, \ \forall f \in \mathbb{R}^\Gamma, \ \forall\, x,y \in \Gamma,$$

which is equivalent to the claim.

Here is another proof of Proposition 3.3, which is inspired from [25]. Fix x, y in Γ and $f \in \mathbb{R}^\Gamma$. Write $B_i = B(x, 2^{-i}d(x,y))$, $i \in \mathbb{N}^*$. One has

$$\begin{aligned}
|f(x) - f_{B_0}| &\le \sum_{i=0}^{+\infty} |f_{B_i} - f_{B_{i+1}}| \\
&\le \sum_{i=0}^{+\infty} {\sum_{B_{i+1}}}' |f - f_{B_i}| \\
&\le C\sum_{i=0}^{+\infty} {\sum_{B_i}}' |f - f_{B_i}| \\
&\le C' \sum_{i=0}^{+\infty} 2^{-i} d(x,y) \left({\sum_{B_i}}' |\nabla f|^p \right)^{1/p}.
\end{aligned}$$

Here one has used (P_p) and (D). Now (LV_Q) implies that

$$\left({\sum_{B_i}}' |\nabla f|^p \right)^{1/p} \le C 2^{\frac{iQ}{p}} d(x,y)^{-\frac{Q}{p}} \||\nabla f|\|_p.$$

Since the series $\sum_{i=0}^{+\infty} 2^{i\left(\frac{Q}{p}-1\right)}$ converges, this yields

$$|f(x) - f_{B(x,d(x,y))}| \leq C' d(x,y)^{(1-\frac{Q}{p})} \| |\nabla f| \|_p.$$

Similarly,

$$|f(y) - f_{B(y,d(x,y))}| \leq C' d(x,y)^{(1-\frac{Q}{p})} \| |\nabla f| \|_p.$$

Finally,

$$|f_{B(x,d(x,y))} - f_{B(y,d(x,y))}| \leq C \sum_{B(x,2d(x,y))}' |f - f_{B(x,2d(x,y))}|,$$

and, using (P_Q) and (LV_Q), one gets

$$\sum_{B(x,2d(x,y))}' |f - f_{B(x,2d(x,y))}| \leq C' d(x,y)^{(1-\frac{Q}{p})} \| |\nabla f| \|_p.$$

The estimate

$$|f(x) - f(y)| \leq C\, d(x,y)^{1-\frac{Q}{p}} \| |\nabla f| \|_p, \ \forall f \in \mathbb{R}^\Gamma, \ \forall\, x, y \in \Gamma$$

is again proved.

This proof has three advantages over the preceding one: it is more direct; it gives a better estimate because one can replace in the RHS $\| |\nabla f| \|_p$ by $\| |\nabla f| \|_{p,B(x,Cd(x,y))}$ (this is explained by Lemma 3.5 below); finally, it only uses Poincaré inequalities around the points x and y, thus it can be completely localized. This partially answers the question asked in [7], p.94. Finally observe that an even shorter proof can be given by starting from (2.2) and using the fact that, under (LV_Q),

$$(M_{Cd(x,y)} |\nabla f|^p)(x) \leq C' \frac{\| |\nabla f| \|_p^p}{d^Q(x,y)}.$$

□

Let us now turn to the sufficiency of (3.1).

Proposition 3.4 *Under (UV_Q), for $p > Q$, the estimate*

$$\delta_p(x,y) \leq C d(x,y)^{1-\frac{Q}{p}}, \ \forall\, x, y \in \Gamma,$$

implies (P_p).

We shall need a lemma, which says that the infinity plays no role.

Lemma 3.5 *Assume (UV_Q). Let $p > Q$. Suppose that there exists $c > 0$ such that*

$$\mathrm{Cap}_p(\{x\}, \{y\}) \geq c\, d(x,y)^{Q-p}, \ \forall\, x, y \in \Gamma, \ x \neq y.$$

Then there exist $C, c' > 0$ such that

$$\mathrm{Cap}_p(\{x\}, \{y\}; B(x, C\, d(x,y))) \geq c'\, d(x,y)^{Q-p}, \ \forall\, x, y \in \Gamma, \ x \neq y.$$

Proof: The proof of Proposition 3.2 shows that there exists C_1 such that

$$\operatorname{Cap}_p(\{x\}, B^c(x,r)) \leq C_1\, r^{Q-p},\ \forall\, x \in \Gamma,\ r > 0.$$

Therefore, for any $c > 0$, there exists C large enough so that

$$\operatorname{Cap}_p(\{x\}, B^c(x, C\, d(x,y))) \leq \frac{c}{2}\, d(x,y)^{Q-p},\ \forall\, x, y \in \Gamma,\ x \neq y.$$

Then the lemma follows from the inequality

$$(*) \qquad \operatorname{Cap}_p(\{x\},\{y\}) \leq \operatorname{Cap}_p(\{x\},\{y\}; B(x, 2C\, d(x,y))) + \operatorname{Cap}_p(\{x\}, B^c(x, C\, d(x,y))),$$

which we are now going to prove.

For $\varepsilon > 0$, let $u_1 \geq 0$ be such that $u_1(x) = 1$, $u_1(y) = 0$, and

$$|||\nabla u_1|||^p_{p, B(x, 2C\, d(x,y))} \leq \operatorname{Cap}_p(\{x\},\{y\}; B(x, 2C\, d(x,y))) + \varepsilon, \tag{3.2}$$

and let $u_2 \geq 0$ be supported in $B(x, C\, d(x,y))$ such that $u_2(x) = 1$ and

$$|||\nabla u_2|||^p_p \leq \operatorname{Cap}_p(\{x\}, B^c(x, C\, d(x,y))) + \varepsilon. \tag{3.3}$$

Then $u = \min(u_1, u_2)$ satisfies $u(x) = 1$, $u(y) = 0$. Also,

$$|||\nabla u|||^p_p \leq |||\nabla u_1|||^p_{p, B(x, 2C\, d(x,y))} + |||\nabla u_2|||^p_p. \tag{3.4}$$

Indeed,

$$|||\nabla u|||^p_p \leq \sum_{\substack{z \in B(x, 2Cd(x,y)) \\ z' \sim z}} |u(z) - u(z')|^p \mu_{zz'} + \sum_{\substack{z \notin B(x, 2Cd(x,y)) \\ z' \sim z}} |u(z) - u(z')|^p \mu_{zz'}. \tag{3.5}$$

Outside $B(x, Cd(x,y))$, u coincides with $u_2 = 0$, therefore $|u(z) - u(z')| = 0$ for $z \notin B(x, 2Cd(x,y))$ and $z' \sim z$ (we can certainly take $C \geq 1$). For all $z, z' \in \Gamma$ such that $z' \sim z$, one sees, using the inequality

$$|\min(a,b) - \min(a',b')| \leq |a - a'| + |b - b'|,$$

that

$$|f(z) - f(z')| \leq |f_1(z) - f_1(z')| + |f_2(z) - f_2(z')|.$$

Applying this to the first term in (3.5) yields (3.4). Then gathering (3.2), (3.3), (3.4), one obtains

$$|||\nabla u|||^p_p \leq \operatorname{Cap}_p(\{x\},\{y\}; B(x, 2C\, d(x,y))) + \operatorname{Cap}_p(\{x\}, B^c(x, C\, d(x,y))) + 2\varepsilon.$$

Since ε was arbitrary, this proves $(*)$, hence the lemma.□

Proof of Proposition 3.4: The upper bound on δ_p is nothing but the assumption of Lemma 3.5, and a reformulation of its conclusion yields

$$|f(y) - f(z)| \leq C\, d(y,z)^{1-\frac{Q}{p}} |||\nabla f|||_{p, B(y, Cd(y,z))},\ \forall f \in \mathbb{R}^\Gamma,\ \forall\, y, z \in \Gamma.$$

Fix $x \in \Gamma$ and $r > 0$, and suppose that $y, z \in B(x,r)$. Then $B(y, Cd(y,z)) \subset B(x, (2C+1)r)$ and

$$|f(y) - f(z)| \leq C' \, r^{1-\frac{Q}{p}} \| |\nabla f| \|_{p, B(x,(2C+1)r)}, \ \forall f \in \mathbb{R}^\Gamma, \ \forall \, x, y \in \Gamma.$$

By taking the average on $z \in B(x,r)$, then the supremum on $y \in B(x,r)$, one gets therefore

$$\sup_{B(x,r)} |f - f_r(x)| \leq C' \, r^{1-\frac{Q}{p}} \| |\nabla f| \|_{p, B(x,(2C+1)r)}, \ \forall f \in \mathbb{R}^\Gamma, \ x, y \in \Gamma, \, r > 0.$$

Now

$$\left(\sum_{B(x,r)} |f - f_r(x)|^p \right)^{1/p} \leq V(x,r)^{1/p} \sup_{B(x,r)} |f - f_r(x)|.$$

Together with (UV_Q), this ends the proof. □

Theorem 3.1 follows from Propositions 3.3 and 3.4.

4 $p = Q$

From now on, we shall assume $Q > 1$. Let us start with a proposition which complements [7], Théorème 4.1, where the cases $p < Q$ and $p > Q$ are treated.

Proposition 4.1 *Under (V_Q), (P_Q) is equivalent to the localized Trudinger-Moser inequality*

$$\sum_{B(x,r)} \exp \left[\left(\frac{c|f - f_r(x)|}{\left(\sum_{B(x,Cr)} |\nabla f|^Q \right)^{1/Q}} \right)^{\frac{Q}{Q-1}} \right] \leq C V(x, 2r),$$

for all $x \in \Gamma$, $r > 0$, $f \in \mathbb{R}^\Gamma$.

Proof: Assume the localized Trudinger-Moser inequality. Since $t \leq e^t$, it implies

$$\left(\sum_{B(x,r)} |f - f_r(x)|^{\frac{Q}{Q-1}} \right)^{\frac{Q-1}{Q}} \leq C' \left(V(x,2r) \right)^{\frac{Q-1}{Q}} \left(\sum_{B(x,Cr)} |\nabla f|^Q \right)^{1/Q}.$$

Now by Hölder

$$\sum_{B(x,r)} |f - f_r(x)| \leq (V(x,2r))^{1/Q} \left(\sum_{B(x,r)} |f - f_r(x)|^{\frac{Q}{Q-1}} \right)^{\frac{Q-1}{Q}},$$

therefore

$$\sum_{B(x,r)} |f - f_r(x)| \leq C' V(x,2r) \left(\sum_{B(x,C'r)} |\nabla f|^Q \right)^{1/Q},$$

and, using (D),

$$\sum_{B(x,r)}^{\prime} |f - f_r(x)| \leq C'' \left(V(x,r)\right)^{1/Q} \left(\sum_{B(x,C'r)}^{\prime} |\nabla f|^Q \right)^{1/Q}.$$

Then (UV_Q) yields

$$\sum_{B(x,r)}^{\prime} |f - f_r(x)| \leq C'' r \left(\sum_{B(x,C'r)}^{\prime} |\nabla f|^Q \right)^{1/Q},$$

which is one of the equivalent forms of (P_Q) (see Section 2.4).

Conversely, (P_Q), (D) and (LV_Q) imply

$$\sum_{x\in\Omega} \exp\left[\left(\frac{c|f(x)|}{\||\nabla f|\|_Q} \right)^{\frac{Q}{Q-1}} \right] \leq C|\Omega|, \ \forall f \in c(\Omega),$$

$\forall\, \Omega$ finite subset of Γ, see Section 2.3. Take $\Omega = B(x, 2r)$ and replace f by $(f - f_r(x))\varphi$, where $\varphi = \frac{1}{r}(r - d(.,B(x,r))_+$. Using (P_Q), one checks that

$$\left(\sum_{B(x,2r+1)} |\nabla([f - f_r(x)]\varphi)|^Q \right)^{1/Q} \leq C \left(\sum_{B(x,C'r)} |\nabla f|^Q \right)^{1/Q}$$

(see [7], pp. 93 and 94 for details). This yields the localized Trudinger-Moser inequality. $\square$

Let us define $\operatorname{diam}(E) = \sup\{d(x,y); x, y \in E\} + 1$; in particular, we consider that a point has diameter one.

The main result in this section is the following.

Theorem 4.2 *Under (V_Q), (P_Q) is equivalent to*

$$\delta_Q(E,F) \leq C \left[\log\left(\frac{d(E,F)}{\min(\operatorname{diam}(E), \operatorname{diam}(F))} + 1 \right) \right]^{\frac{Q-1}{Q}}, \tag{4.1}$$

for all E, F disjoint connected subsets of Γ.

Note that the result of Proposition 2.6 can be reformulated in this case as

$$\delta_Q(E,F,B(x,Cr)) \leq \frac{Cr}{\min(|E|^{1/Q}, |F|^{1/Q})}, \ \forall\, x \in \Gamma,\ r > 0,\ E, F \text{ disjoint subsets of } B(x,r),$$

where $\delta_Q(E,F,B(x,Cr))$ has the obvious definition. The latter condition implies

$$\delta_Q(E,F) \leq C\, \frac{d(E,F) + \operatorname{diam}(E) + \operatorname{diam}(F)}{\min(|E|^{1/Q}, |F|^{1/Q})}, \ \forall\, x \in \Gamma,\ r > 0,\ E, F \text{ disjoint subsets of } B(x,r).$$

Under (LV_Q), (4.1) is strictly stronger, but it is required only for connected sets. Compare also with (4.10) below.

We shall first prove the necessity of condition (4.1). For this we need a lemma.

Lemma 4.3 *Suppose that (P_Q), (LV_Q) and (D) hold. Given $c>0$, there exist $C, c'>0$ such that the following holds. Suppose $E\subset B(x,r)$ is connected with* $\mathrm{diam}(E)\geq cr$, *$f(y)=0$ for all $y\in E$, and $f_{B(x,r)}\geq 1$. Then*

$$\sum_{B(x,Cr)} |\nabla f|^Q \geq c'.$$

We shall prove in fact a more general version of this fact, which will be useful in Section 6 below. For this purpose, let us recall the notion of Hausdorff content (for details, see for instance [30], chapter 4). Given $\lambda>0$ and $E\subset\Gamma$, we set

$$H^\lambda_\infty(E)=\inf\{\sum_i r_i^\lambda;\ r_i\geq 1,\ E\subset \cup_i B(x_i,r_i)\}.$$

(note that the r_i are required to be large enough, a modification due to the discreteness of space). Then H^λ_∞ is an (outer) measure and always $H^\lambda_\infty(E)\leq \mathrm{diam}(E)^\lambda$.

Lemma 4.4 *Let $p\in[1,+\infty[$, $Q>1$, and $\lambda>Q-p$. Suppose that (P_p), (LV_Q) and (D) hold. Then there exist $C,c>0$ such that the following holds. Suppose $E\subset B(x,r)$, $f(y)=0$ for all $y\in E$, and $f_{B(x,r)}\geq 1$. Then*

$$r^{\lambda+p-Q}\sum_{B(x,2Cr)}|\nabla f|^p\geq cH^\lambda_\infty(E).$$

Proof: Let $y\in E$, f as above, and let $k\geq 1$ be the least integer with $2^k\geq 2\mathrm{diam}(E)$. Then the triangle inequality and (P_p) give, as in the second proof of Proposition 3.3,

$$|f(y)-f_{2^k}(y)|\leq C\sum_0^k 2^i\left(\sum_{B(y,C'2^i)}{}' |\nabla f|^p\right)^{1/p}.$$

Assume first that $|f_{2^k}(y)|\geq 1/3$ for each $y\in E$. Then (LV_Q) and the above estimate give

$$1/3\leq C\sum_0^k 2^{i(1-Q/p)}\||\nabla f|\|_{p,B(y,C'2^i)}. \tag{4.2}$$

Rewrite (4.2) as

$$1/3\leq C\sum_0^k 2^{i(\lambda+p-Q)/p}2^{-i\lambda/p}\||\nabla f|\|_{p,B(y,C'2^i)}. \tag{4.3}$$

Let M be such that

$$\sum_0^k 2^{i(\lambda+p-Q)/p}\leq M2^{k(\lambda+p-Q)/p},\ \forall\, k\in\mathbb{N}^*. \tag{4.4}$$

If one had

$$2^{-i\lambda/p}\||\nabla f|\|_{p,B(y,C'2^i)}<c2^{-k(\lambda+p-Q)/p},$$

for every i, $0\leq i\leq k$, with $c=(3CM)^{-1}$, then (4.3) would be in contradiction with (4.4), therefore, for some i, $0\leq i\leq k$,

$$2^{k(\lambda+p-Q)/p}\||\nabla f|\|_{p,B(y,C'2^i)}\geq c2^{i\lambda/p}.$$

In other terms, $\forall\, y \in E$, $\exists\, r_y > 0$ such that

$$2^{k(\lambda+p-Q)} |||\nabla f|||^p_{p,B(y,r_y)} \geq c r_y^{\lambda}.$$

By the usual covering lemma, pick up a collection of disjoint balls $B_i = B(y_i, r_{y_i})$ such that $E \subset \bigcup_i 5B_i$. Then

$$2^{k(\lambda+p-Q)} |||\nabla f|||^p_{p,B(x,C'r)} \geq 2^{k(\lambda+p-Q)} \sum_i |||\nabla f|||^p_{p,B_i} \geq c \sum_i r_i^{\lambda},$$

as desired. We may thus assume that $f_{2^k}(y) \leq 1/3$ for some $y \in E$. As above,

$$1/9 \leq |f_{2^k}(y) - f_{B(x,r)}| \leq C r^{(p-Q)/p} |||\nabla f|||_{p,B(y,2C'r)}.$$

The claim follows. □

Lemma 4.3 obviously follows from Lemma 4.4: take $p = Q$, $\lambda = 1$ and notice that, for a connected E,

$$H^1_{\infty}(E) \geq \mathrm{diam}(E)/2.$$

Proposition 4.5 *(P_Q), (LV_Q) and (D) imply* (4.1).

Proof: Let us assume (LV_Q) and (P_Q). Suppose for example that $\mathrm{diam}(E) \leq \mathrm{diam}(F)$. Let $B(x,r)$ be a ball with $x \in E$ that contains E, such that $\mathrm{diam}(\tilde{F}) \geq \mathrm{diam}(E)$, where $\tilde{F} \subset F \cap B(x,r)$ is a connected set and with $r \leq 2(\mathrm{diam}(E) + d(E,F))$. Pick a minimal path P that joins x to $\tilde{F}$ in $B(x,r)$. Then the length of P is at most r. Let $x_1 \in P \cap E$ and define $r_1 = \mathrm{diam}(E)$ and $B_1 = B(x_1, r_1)$. Set $\lambda = (2C'+1)/2C'$, where C' is the constant in (P_Q). Trace along P beginning from x_1 until P exits B_1 for the last time, pick the corresponding point x_2 from P, and define $r_2 = \lambda r_1$ and $B_2 = B(x_2, r_2)$. We continue inductively. Assuming that x_i, r_i, B_i have been defined, we trace along P beginning from x_1 to see if P exits B_i. If it does not, we stop the process. Otherwise, we pick the corresponding point x_{i+1} from P, and define $r_{i+1} = \lambda r_i$ and $B_{i+1} = B(x_{i+1}, r_{i+1})$. As the length of P does not exceed n, we see that the process terminates before $\lambda^{i+1}\mathrm{diam}(E) \geq r$. Let B_k be the last ball that we have constructed.

For every $f \in \mathbb{R}^{\Gamma}$ satisfying $f|_E = 1$ and $f|_F = 0$, we have to prove the estimate

$$k^{(Q-1)/Q} |||\nabla f|||_Q \geq c \tag{4.5}$$

for some $c > 0$. Indeed, this shows that

$$\delta_Q(E,F) \leq C k^{(Q-1)/Q},$$

and since $\lambda^{k+1}\mathrm{diam}(E) \leq r \leq 2(\mathrm{diam}(E) + d(E,F))$, (4.1) follows.

We have

$$\begin{aligned}
|f_{B_k} - f_{B_1}| &\le \sum_{i=1}^{k-1} |f_{B_i} - f_{B_{i+1}}| \\
&\le \sum_{i=1}^{k-1} \sideset{}{'}\sum_{B_i} |f - f_{B_{i+1}}| \\
&\le C \sum_{i=1}^{k-1} \sideset{}{'}\sum_{B_{i+1}} |f - f_{B_{i+1}}| \\
&\le C \sum_{i=2}^{k} \lambda^i \mathrm{diam}(E) \left(\sideset{}{'}\sum_{C'B_i} |\nabla f|^Q \right)^{1/Q} .
\end{aligned}$$

Here we used (P_Q) and (D). Now (LV_Q) implies that

$$\lambda^i \mathrm{diam}(E) \left(\sideset{}{'}\sum_{C'B_i} |\nabla f|^Q \right)^{1/Q} \le C' \left(\sum_{C'B_i} |\nabla f|^Q \right)^{1/Q} ,$$

hence

$$|f_{B_k} - f_{B_1}| \le C \sum_{i=2}^{k} \left(\sum_{C'B_i} |\nabla f|^Q \right)^{1/Q} . \tag{4.6}$$

Notice that there is a universal constant M depending only on the constant λ so that no more than M of the above balls B_i can intersect at a given point. Thus, using the Hölder inequality in (4.6), we conclude that

$$|f_{B_k} - f_{B_1}| \le C'' k^{(Q-1)/Q} \left(\sum_{B(x,C'r)} |\nabla f|^Q \right)^{1/Q} . \tag{4.7}$$

We pick the first point $y_1 \in P$ that belongs to $\tilde{F}$ and repeat the construction by using the subpath of P from y_1 to x_1. If we denote the resulting balls by B'_i, the above argument gives

$$|f_{B'_{k'}} - f_{B'_1}| \le C'' (k')^{(Q-1)/Q} \left(\sum_{B(x,C'r)} |\nabla f|^Q \right)^{1/Q} . \tag{4.8}$$

Using (P_Q), doubling and (LV_Q), we see that

$$|f_{B_k} - f_{B'_{k'}}| \le \left(\sum_{B(x,Cr)} |\nabla f|^Q \right)^{1/Q} . \tag{4.9}$$

If $f_{B_1} \ge 1/6$ or if $f_{B'_1} \le 5/6$, then the desired estimate (4.5) follows from Lemma 4.3. If $|f_{B_k} - f_{B'_{k'}}| \ge 1/3$, (4.9) gives the claim. Thus we may assume that $f_{B_k} \ge 1/3$ or $f_{B'_{k'}} \le 2/3$, and that $f_{B_1} \le 1/6$, $f_{B'_1} \ge 5/6$. The use of inequalities (4.7), (4.8) then completes the proof.□

Before we turn to the other part of the proof of Theorem 4.2, let us give a slightly different version of Proposition 4.5.

Proposition 4.6 *(P_Q) and (V_Q) imply*

$$\delta_Q(E,F;B(x,C'r)) \leq C\left[\log\left(\frac{r}{\min(|E|^{1/Q},|F|^{1/Q})}\right)\right]^{\frac{Q-1}{Q}}, \tag{4.10}$$

for some $C>0$, $C'\geq 1$, and all $x\in\Gamma$, $r>0$, and E,F disjoint subsets of $B(x,r)$.

Proof: Let $B(x,r)$ be a ball that contains E and F. According to Proposition 4.1, (P_Q) and (V_Q) imply

$$\sum_{B(x,r)} \exp\left[\left(\frac{c|f-f_r(x)|}{\left(\sum_{B(x,C'r)}|\nabla f|^Q\right)^{1/Q}}\right)^{\frac{Q}{Q-1}}\right] \leq CV(x,2r).$$

Let f be a function on Γ such that $f\mid_E = 1$, $f\mid_F = 0$.

Now, either $|f_r(x)-1|\geq 1/2$ or $|f_r(x)|\geq 1/2$. Suppose for example that one is in the first case, otherwise one exchanges the roles of E and F. Then

$$|E|\exp\left[\left(\frac{c/2}{\left(\sum_{B(x,C'r)}|\nabla f|^Q\right)^{1/Q}}\right)^{\frac{Q}{Q-1}}\right] \leq \sum_E \exp\left[\left(\frac{c|f-f_r(x)|}{\left(\sum_{B(x,C'r)}|\nabla f|^Q\right)^{1/Q}}\right)^{\frac{Q}{Q-1}}\right] \leq CV(x,2r).$$

Therefore

$$\left(\sum_{B(x,C'r)}|\nabla f|^Q\right)^{1/Q} \geq \frac{c/2}{\left(\log C\frac{V(x,2r)}{|E|}\right)^{\frac{Q-1}{Q}}},$$

and

$$\delta_Q(E,F;B(x,C'r)) \leq C\left(\log\frac{V(x,2r)}{|E|}+1\right)^{\frac{Q-1}{Q}} \leq C'\left(\log\frac{r^Q}{|E|}+1\right)^{\frac{Q-1}{Q}}$$

by (UV_Q). This gives the claimed estimate. □

To prove the sufficiency of condition (4.1), we need as in Section 3 a lemma that enables us to eliminate the infinity.

Lemma 4.7 *Assume that Γ satisfies (UV_Q). Let $E,F\subset B(x,r)$ be disjoint subsets with*

$$\mathrm{Cap}_Q(E,F)\geq c>0.$$

Then there exists $C\geq 1$ only depending on c,Q and the constant in (UV_Q) such that

$$\mathrm{Cap}_Q(E,F;B(x,C\,r)) \geq \frac{1}{2}\mathrm{Cap}_Q(E,F).$$

Proof: For $j \geq 2r$, define

$$f_{(j)}(z) = \min\left(1, \left(\log\frac{j}{r}\right)^{-1}\left(\log\frac{d(z,x)}{r}\right)_+\right)$$

for $z \neq x$, and $f_{(j)}(x) = 0$. Note that $f_{(j)} \equiv 0$ on $B(x,r)$ and $f_{(j)} \equiv 1$ on $B^c(x,j)$. From (UV_Q) we obtain, by splitting the exterior of the ball $B(x,r)$ into annuli $B(x, 2^{k+1}r) \setminus B(x, 2^k r)$,

$$|||\nabla f_{(j)}|||_Q^Q \leq C_0\left(\log\frac{j}{r}\right)^{1-Q},$$

where C_0 depends only on Q and the constant in (UV_Q). Fix j large enough so that

$$C_0\left(\log\frac{j}{r}\right)^{1-Q} \leq c/2.$$

We have shown that

$$\mathrm{Cap}_Q(B(x,r), B^c(x,Cr)) \leq c/2$$

whenever $C \geq \frac{j}{r}$. The claim then follows from the inequality

$$\mathrm{Cap}_Q(E,F) \leq \mathrm{Cap}_Q(E,F; B(x, 2C\,r)) + \mathrm{Cap}_Q(B(x,r), B^c(x, C\,r))$$

whose proof is identical to that of inequality $(*)$ in the proof of Lemma 3.5.□

At this point, let us mention the related reference [37], where it is proved that under (UV_Q), Γ is Q-parabolic, that is $\mathrm{Cap}_Q(E,\emptyset) = 0$ for every finite E.

Now the converse part of Theorem 4.2 will be a consequence of the following result, which we state for all values of p in order to be able to use it as well in Section 5.

Proposition 4.8 *Under (UV_Q) and (D), the estimate*

$$\mathrm{Cap}_p(E,F; B(x,Cr)) \geq c\,r^{Q-p},$$

for some $p \in [1,+\infty)$, $C \geq 1$, $c > 0$, for all $x \in \Gamma$, $r > 0$, E, F disjoint connected subsets of $B(x,r)$ with $\min(\mathrm{diam}(E), \mathrm{diam}(F)) \geq r/100$, *implies (P_p).*

Proof: By Lemma 2.4 it suffices to show that

$$|f(x) - f(y)| \leq C\,d(x,y)\left(M_{Cd(x,y)}|\nabla f|^p(x) + M_{Cd(x,y)}|\nabla f|^p(y)\right)^{1/p},$$

$\forall f \in \mathbb{R}^\Gamma$, $\forall x, y \in \Gamma$.

Let f, x, y be as above. Without loss of generality, assume that $f(x) = 1, f(y) = 0$. Let $k \in \mathbb{N}$ be such that $2^{k+1} < d(x,y) \leq 2^{k+2}$.

Suppose first that

$$\sum_P |\nabla f|_P \geq 1/4$$

for each path P that joins $B(x,2^{k-1})$ to $B(y,2^{k-1})$ in $B(x,C2^{k+3})$. Then Proposition 2.2 and the assumed lower bound on the capacity yield

$$4^p|||\nabla f|||^p_{p,B(x,C2^{k+3})} \geq \mathrm{Cap}_p(B(x,2^{k-1}),B(y,2^{k-1});B(x,C2^{k+3})) \geq c(2^{k+3})^{Q-p},$$

hence

$$8^p 2^{-(k+3)Q}|||\nabla f|||^p_{p,B(x,C2^{k+3})} \geq c2^{-(k+2)p},$$

that is, by (UV_Q),

$$\frac{\sum_{B(x,C2^{k+3})}|\nabla f|^p}{V(x,C2^{k+3})} \geq c'2^{-(k+2)p},$$

and by the choice of k,

$$M_{4Cd(x,y)}|\nabla f|^p(x) \geq c'd(x,y),$$

hence the claim.

Otherwise we can find a path P_1 joining $B(x,2^{k-1})$ to $B(y,2^{k-1})$ in $B(x,C2^{k+3})$ so that $\sum_{P_1}|\nabla f|_{P_1} \leq 1/4$. We repeat the argument with $B(x,2^{k-1})$ replaced by $B(x,2^{k-2})$ and $B(y,2^{k-1})$ replaced by a subpath of P_1 that joins the boundary of $B(x,2^{k-1})$ to the boundary of $B(x,2^k)$.

If $\sum_P|\nabla f|_P \geq 1/16$, for each path P that joins $B(x,2^{k-2})$ to the given subpath of P_1 inside $B(x,C2^k)$, we reason as above to see that

$$16^p|||\nabla f|||^p_{p,B(x,C2^k)} \geq c(2^k)^{Q-p}$$

hence

$$4^p 2^{-kQ}|||\nabla f|||^p_{p,B(x,C2^k)} \geq c2^{-(k+2)p},$$

and the claim follows. Otherwise $\sum_P|\nabla f|_P \leq 1/16$ for some such path P; by argueing in the same way around y, either the claim is proved or we find a path P_2 that joins $B(x,2^{k-2})$ to $B(y,2^{k-2})$ with $\sum_{P_2}|\nabla f|_{P_2} \leq 1/4+1/8$.

We continue inductively: at stage $1 \leq j$ either

$$2^{(j+3)p}|||\nabla f|||^p_{p,B(z,C2^{k-j+1})} \geq c(2^{k-j+1})^{Q-p}$$

for $z=x$ or for $z=y$ and the claim follows or we obtain a path P_j that joins $B(x,2^{k-j-1})$ to $B(y,2^{k-j-1})$ with

$$\sum_{P_j}|\nabla f|_{P_j} \leq 1/4+1/8+...+2^{-j-2}.$$

By letting $j=k$ the path P_j joins x to y and the above sum is no more than $1/2$ which is impossible as $f(x)=1$ and $f(y)=0$. Thus the first alternative holds for some j and the proof is complete. □

We can now state:

Proposition 4.9 *Under (UV_Q) and (D), (4.1) implies (P_Q).*

Proof: Let $x \in \Gamma$, $r > 0$, and E, F disjoint connected subsets of $B(x,r)$ with

$$\min(\mathrm{diam}(E), \mathrm{diam}(F)) \geq r/100.$$

By (4.1),

$$\mathrm{Cap}_Q(E,F) \geq \frac{c}{\left[\log\left(\frac{d(E,F)}{\min(\mathrm{diam}(E),\mathrm{diam}(F))}+1\right)\right]^{Q-1}} \geq \frac{c}{\left[\log\left(\frac{d(E,F)}{r}+1\right)\right]^{Q-1}},$$

and finally, since $d(E,F) \leq r$,

$$\mathrm{Cap}_Q(E,F) \geq c'.$$

Thus Lemma 4.7 applies, and

$$\mathrm{Cap}_Q(E,F;B(x,Cr)) \geq c'' > 0.$$

One then concludes by using Proposition 4.8 in the case $p = Q$. □

Theorem 4.2 follows from Propositions 4.5 and 4.9. One can see easily that a similar characterization does not hold if one replaces connected sets E, F by points x, y. Indeed, there exist graphs satisfying (V_Q) and

$$\delta_Q(x,y) \leq C\left(\log d(x,y)+1\right)^{\frac{Q-1}{Q}}, \ \forall\, x,y \in \Gamma, x \neq y$$

but not (P_Q). Consider the graph Γ made of two copies of $\mathbb{Z}^Q$ glued by a single edge. The volume growth of Γ is polynomial of exponent Q. The inequality

$$\delta_Q(x,y) \leq C(\log d(x,y)+1)^{\frac{Q-1}{Q}}, \ \forall\, x,y \in \Gamma, x \neq y,$$

which is nothing but (2.4), follows from its counterpart on $\mathbb{Z}^Q$. However (P_Q) is false on Γ.

Finally, we refer the reader to [5] for more consequences of (P_Q).

5 $1 \leq p < Q$

It is well-known that, if $1 \leq p < +\infty$, the Sobolev inequality

$$\|f\|_{\frac{pQ}{Q-p}} \leq C\||\nabla f|\|_p, \ \forall f \in c_0(\Gamma),$$

is equivalent to the estimate

$$\mathrm{Cap}_p(E) \geq c\,|E|^{\frac{Q-p}{Q}}, \ \forall\, E \subset \Gamma,$$

where

$$\mathrm{Cap}_p(E) = \inf\{\||\nabla f|\|_p^p; f \in c_0(\Gamma), f\,|_E \geq 1\}$$

(see [31], [27], [2], Section 10.1, and [18]).

Let us start by noticing that global Sobolev inequalities are equivalent to similar estimates for pairs of sets. Our task will then be to improve these estimates if (P_p) holds; we will get this way a characterisation of (P_p) only for a limited range of p's.

One says that Γ satisfies a global Sobolev inequality at the level p, $1 \leq p < Q$, if there exists C such that, for every function f on Γ such that $|\nabla f| \in L^p$, there exists a number $c(f)$ such that

$$\|f - c(f)\|_{\frac{pQ}{Q-p}} \leq C \| |\nabla f| \|_p \tag{5.1}$$

(see [35] and references therein).

Proposition 5.1 *Let $p \in [1, Q[$. The global Sobolev inequality* (5.1) *is equivalent to the estimate*

$$\mathrm{Cap}_p(E, F) \geq c \min(|E|, |F|)^{\frac{Q-p}{Q}}, \tag{5.2}$$

for all disjoint subsets E, F of Γ.

Proof: Assume (5.1). Let f be a function on Γ such that $\||\nabla f|\|_p < +\infty$, $f\,|_E = 1$, $f\,|_F = 0$. Suppose that $|c(f) - 1| \geq 1/2$; to treat the case $|c(f)| \geq 1/2$, one exchanges the roles of E and F. Then

$$\||\nabla f|\|_p^p \geq c \|f - c(f)\|_{\frac{pQ}{Q-p}}^p \geq c \left(\sum_E |f - c(f)|^{\frac{pQ}{Q-p}} \right)^{\frac{Q-p}{Q}} \geq c 2^{-p} |E|^{\frac{Q-p}{Q}}.$$

The capacity estimate (5.2) follows.

Conversely, let f be a function on Γ such that $\||\nabla f|\|_p < +\infty$. Given $m < M$, set

$$f_{m,M}(x) = \min(M, \max\{f(x), m\})$$

and

$$g_{m,M} = (f_{m,M}(x) - m)/(M - m).$$

Then $g_{m,M}(x) = 0$ in $\{x;\ f(x) \leq m\}$, and $g_{m,M}(x) = 1$ in $\{x;\ f(x) \geq M\}$. Because

$$\||\nabla g_{m,M}|\|_p \leq (M - m)^{-1} \||\nabla f_{m,M}|\|_p,$$

(5.2) implies that

$$c \min(|E_m|, |E^M|)^{\frac{Q-p}{Q}} \leq (M - m)^{-p} \sum |\nabla f_{m,M}|^p < \infty, \tag{5.3}$$

where $E_m = \{x;\ f(x) \leq m\}$ and $E^M = \{x;\ f(x) \geq M\}$. Now, fix $x_0 \in \Gamma$, choose first r large enough so that

$$\sum_{B^c(x_0, r-1)} |\nabla f|^p < c |B(x_0, r)|^{\frac{Q-p}{Q}}, \tag{5.4}$$

and then M_0 so that $f(x) > M_0$ for every $x \in B(x_0, r)$, that is $B(x_0, r) \subset E^{M_0}$. Setting $m_0 = M_0 - 1$, (5.3) yields

$$c \min(|E_{m_0}|, |B(x_0, r)|)^{\frac{Q-p}{Q}} \leq \sum_{B^c(x_0, r-1)} |\nabla f|^p,$$

hence, because of (5.4),

$$c|E_{m_0}|^{\frac{Q-p}{Q}} \leq \sum_{B(x_0,r-1)^c} |\nabla f|^p.$$

We conclude that $|\{x; f(x) \leq m\}| < \infty$ whenever $m \leq m_0$, therefore, since Γ is infinite, $|\{x; f(x) > m\}| = \infty$ whenever $m \leq m_0$.

Set

$$c(f) = \sup\{t;\ |\{x;\ f(x) > t\}| = \infty\}.$$

It follows from the above that this definition makes sense. Also, one can show as in the previous paragraph that $|E^M| < \infty$ for M large enough, therefore $c(f) < \infty$. Moreover, by the definition,

$$|\{x;\ f(x) \leq c(f) + \epsilon\}| = \infty$$

for each $\epsilon > 0$. Notice also that

$$|\{x;\ f(x) \geq c(f) - \epsilon\}| = \infty.$$

Now

$$\begin{aligned}\sum |f - c(f)|^{pQ/(Q-p)} &\leq \sum_{j=-\infty}^{\infty} \sum_{\{x;\, 2^j \leq |f(x)-c(f)| \leq 2^{j+1}\}} |f - c(f)|^{pQ/(Q-p)} \\ &\leq \sum_{j=-\infty}^{\infty} 2^{(j+1)pQ/(Q-p)} |\{x;\ |f(x) - c(f)| \geq 2^j\}|.\end{aligned}$$

Consider the decomposition

$$\{x;\ |f(x) - c(f)| \geq 2^j\} = A_j^+ \cup A_j^-,$$

where $A_j^+ = \{x;\ f(x) - c(f) \geq 2^j\}$ and A_j^- is the residual set. Because

$$|\{x;\ f(x) - c(f) \leq 2^{j-1}\}| = \infty,$$

(5.3) yields the estimate

$$c|A_j^+|^{\frac{Q-p}{Q}} \leq 2^{-(j-1)p} \sum |\nabla f_{(j)}|^p,$$

where $f_{(j)} = f_{c(f)+2^{j-1},\ c(f)+2^{j-1}}$. An analogous estimate follows for $|A_j^-|$. Thus

$$|\{x;\ |f(x) - c(f)| \geq 2^j\}| \leq C 2^{-jpQ/(Q-p)} \left(\sum |\nabla f_{(j)}|^p\right)^{\frac{Q}{Q-p}},$$

and, summing in j,

$$\begin{aligned}\sum |f - c(f)|^{pQ/(Q-p)} &\leq C' \sum_{j=-\infty}^{\infty} \left(\sum |\nabla f_{(j)}|^p\right)^{\frac{Q}{Q-p}} \\ &\leq C' \left(\sum_{j=-\infty}^{\infty} \sum |\nabla f_{(j)}|^p\right)^{\frac{Q}{Q-p}}.\end{aligned}$$

By [2], Section 7, or [22], Lemma 12.1,

$$\sum \sum_{j=-\infty}^{\infty} |\nabla f_{(j)}|^p \leq C \sum |\nabla f|^p$$

and (5.1) is proved. □

Remark: A modification of the above proof yields the converse part in Proposition 2.6: one chooses $c(f)$ so that both $|\{y \in B(x,r);\ f(y) \leq c(f)\}| \geq |B(x,r)|/2$ and $|\{y \in B(x,r);\ f(y) \geq c(f)\}| \geq |B(x,r)|/2$.

In terms of δ_p, the conclusion of the above proposition is that

$$\delta_p(E,F) \leq \frac{C}{\min(|E|,|F|)^{\frac{Q-p}{pQ}}}.$$

This means that two sets are δ_p-close as soon as they are both big enough, but their relative distance d plays no role, contrary to what happens when $p \geq Q$. The estimates we are going to obtain now are of the same style, even though the measure of the sets is replaced by their diameter. This is why we shall express them in terms of Cap_p rather than δ_p.

5.1 $Q-1 < p < Q$

Theorem 5.2 *Let $p \geq 1$. Assume $Q-1 < p < Q$. Under (V_Q), (P_p) holds if and only if, for some $C \geq 1$ and $c > 0$,*

$$\mathrm{Cap}_p(E,F;B(x,Cr)) \geq c \min(\mathrm{diam}(E), \mathrm{diam}(F))^{Q-p}, \tag{5.5}$$

for all $x \in \Gamma$, $r > 0$, E,F disjoint connected subsets of $B(x,r)$.

Remarks:

-Since $\mathrm{diam}(E), \mathrm{diam}(F) \leq 2r$ and, by (UV_Q), $|E| \leq C(\mathrm{diam}(E))^Q$, $|F| \leq C(\mathrm{diam}(F))^Q$, condition (5.5) is apparently stronger than (2.3). Note however that it is restricted to connected sets.

- According to Proposition 5.1, the global Sobolev inequality (5.1) only gives, if E and F are connected,

$$\mathrm{Cap}_p(E,F) \geq c \min(|E|,|F|)^{\frac{Q-p}{Q}} \geq c \min(\mathrm{diam}(E), \mathrm{diam}(F))^{\frac{Q-p}{Q}}.$$

Conversely, under (UV_Q), the inequality

$$\mathrm{Cap}_p(E,F) \geq c \min(\mathrm{diam}(E), \mathrm{diam}(F))^{Q-p},$$

which follows from (5.5), implies

$$\mathrm{Cap}_p(E,F) \geq c' \min(|E|,|F|)^{\frac{Q-p}{Q}},$$

for connected E, F. This is consistent with the fact that (P_p) together with (V_Q) implies (5.1) (see [35]).

- In the above condition, one computes the capacity in the ball $B(x, Cr)$ instead in all of Γ. This is essential as the graph obtained from $\mathbb{Z}^2$ by removing all the vertices $(m, 0)$, $m \geq 0$ that are not of the form $(2^n, 0)$ and the corresponding edges satisfies the above capacity condition provided the capacities are computed over all of Γ but (P_p) fails on Γ for all $1 \leq p < 2$. To see that (P_p) fails on Γ, consider balls of the form $B((2^n, 0), 2^{n-1} - 1)$. Such a ball consists of an upper part and a lower part, both connected and of essentially diameter 2^{n-1} and volume $2^{2(n-1)}$, connected along a single edge. For the function f that equals zero on the lower part together with the connecting edge and equals one on the upper part, we clearly have that the left-hand side in (2.1) is bounded from below by a constant, whereas the right-hand side is at most a constant multiple of $2^{(1-2/p)(n-1)}$.

To see that the capacity condition holds with $B(x, Cr)$ replaced by all of the graph requires some work. The essential difficulty is to obtain the estimate for the case when E is in the upper part and F in the lower. To handle this, one joins points in E and F by a sequence of balls whose sizes increase geometrically when one moves away from E, F by travelling through the part of the graph left from $(0, 0)$. The required computations are similar to those in the proof of Proposition 5.3 below.

By modifying the proof of Lemma 4.4, one can also see that (5.2) holds on this graph, which yields therefore an example satisfying (V_Q) for $Q = 2$, the global Sobolev inequality (5.1) for $1 \leq p < Q$, but not (P_p). The main result in [35] has therefore no converse.

The sufficiency of condition (5.5) for (P_p) follows directly from Proposition 4.8. The necessity can be proved along the same lines as Lemma 4.3.

Proposition 5.3 *If $p \geq 1$, $Q - 1 < p < Q$, and (P_p), (LV_Q) and (D) hold, then there exist $C \geq 1$, $c > 0$ such that*

$$\operatorname{Cap}_p(E, F; B(x, Cr)) \geq c \min(\operatorname{diam}(E), \operatorname{diam}(F))^{Q-p},$$

for all $x \in \Gamma$, $r > 0$, E, F disjoint connected subsets of $B(x, r)$.

Proof: Without loss of generality we may assume that $\operatorname{diam}(E) = \operatorname{diam}(F)$. Let $y \in \Gamma$ and let $k \geq 1$ be the least integer with $2^k \geq 2\operatorname{diam}(E)$. Then the triangle inequality and (P_p) give, as in the second proof of Proposition 3.3,

$$|f(y) - f_{2^k}(y)| \leq C \sum_0^k 2^i \Big(\sum_{B(y,2^i)}' |\nabla f|^p \Big)^{1/p}.$$

Let f be such that $f|_E = 0$ and $f|_F = 1$.

Assume first that $|f_{2^k}(y)| \geq 1/3$ for each $y \in E$. Then (LV_Q) and the above estimate give

$$1/3 \leq C \sum_0^k 2^{i(1-(Q/p))} \| |\nabla f| \|_{p, B(y, 2^i)}. \tag{5.6}$$

Set $\varepsilon = 1 - (Q/p) + (1/p) = (p+1-Q)/p > 0$. Rewrite (5.6) as

$$1/3 \leq C \sum_0^k 2^{i\varepsilon} 2^{-i/p} |||\nabla f|||_{p,B(y,2^i)}. \tag{5.7}$$

Let $M(\varepsilon)$ be such that

$$\sum_0^k 2^{\varepsilon i} \leq M(\varepsilon) 2^{\varepsilon k}, \ \forall\, k \in \mathbb{N}^*. \tag{5.8}$$

If one had

$$2^{-i/p} |||\nabla f|||_{p,B(y,2^i)} < c 2^{-\varepsilon k},$$

for every i, $0 \leq i \leq k$, with $c = (3CM(\varepsilon))^{-1}$, then (5.7) would be in contradiction with (5.8), therefore, for some i, $0 \leq i \leq k$,

$$2^{\varepsilon k} |||\nabla f|||_{p,B(y,2^i)} \geq c 2^{i/p}.$$

In other terms, $\forall\, y \in E$, $\exists\, r_y > 0$ such that

$$C 2^{(p+1-Q)k} |||\nabla f|||^p_{p,B(y,r_y)} \geq r_y.$$

By the usual covering lemma, pick up a collection of disjoint balls $B_i = B(y_i, r_{y_i})$ such that $E \subset \bigcup_i 5B_i$. Then

$$C 2^{(p+1-Q)k} |||\nabla f|||^p_{p,E} \geq 2^{(p+1-Q)k} \sum_i |||\nabla f|||^p_{p,B_i} \geq c \sum_i \operatorname{diam}(B_i) \geq c' \operatorname{diam}(E),$$

since E is connected.

The desired estimate follows in this case as $2^k \leq 4\operatorname{diam}(E)$. The case where $|f_{2^k}(z)| \geq 2/3$ for every $z \in F$ can be treated similarly.

We may thus assume that $f_{2^k}(y) \leq 1/3$ and $f_{2^k}(z) \geq 2/3$ for some $y \in E$ and some $z \in F$. Suppose $f_{2^\ell}(y) \geq 4/9$, where ℓ is the least integer with $2^\ell \geq r$. As above,

$$1/9 \leq |f_{2^k}(y) - f_{2^\ell}(y)| \leq C \sum_{i=k+1}^{\ell} 2^{i(1-Q/p)} |||\nabla f|||_{p,B(y,2^i)} \leq C' 2^{k(1-Q/p)} |||\nabla f|||_{p,B(y,2^\ell)}$$

where the last inequality uses $Q/p > 1$. The desired estimate follows from this inequality. The case where $f_{2^\ell}(z) \leq 5/9$ is similar.

We are only left with the case where $f_{2^\ell}(y) \leq 4/9$ for some $y \in E$ and $f_{2^\ell}(z) \geq 5/9$ for some $z \in F$. Using (D) and (P_p) we obtain

$$\begin{aligned}
1/9 \leq |f_{2^\ell}(z) - f_{2^\ell}(y)| &\leq \frac{1}{V(z,2^\ell)} \sum_{B(z,2^\ell)} |f - f_{3r}(x)| + \frac{1}{V(y,2^\ell)} \sum_{B(y,2^\ell)} |f - f_{3r}(x)| \\
&\leq C \sum_{B(x,3r)}{}' |f - f_{3r}(x)| \\
&\leq C' r \left(\sum_{B(x,3r)}{}' |\nabla f|^p \right)^{1/p}.
\end{aligned}$$

Here we used the fact that $B(z, 2^\ell) \cup B(y, 2^\ell) \subset B(x, 3r)$. It then follows from (LV_Q) that

$$1/9 r^{Q-p} \leq C'' \| |\nabla f| \|_{p, B(x,3r)},$$

and the claim is once again proved. □

5.2 $p = Q - 1$

Here we assume $Q \geq 2$. The preceding theorem does not cover the case $p = Q - 1$. In fact, the lower bound

$$\mathrm{Cap}_{Q-1}(E, F) \geq c \min(\mathrm{diam}(E), \mathrm{diam}(F))$$

is false in $X = \mathbb{Z}^Q$, $Q \geq 3$.

Indeed, let E be the interval $[0, M]$ in $\mathbb{Z} \subset \mathbb{Z}^Q$. We define $f(x) = (\log M)^{-1} \log(d(x, E))$, when $1 \leq d(x, E) \leq M$, $f(x) = 0$, when $x \in E$, and $f(x) = 1$, when $d(x, E) > M$. Then a straightforward calculation gives $\| |\nabla f| \|_{Q-1}^{Q-1} \leq CM(\log M)^{2-Q}$, hence

$$\mathrm{Cap}_{Q-1}(E, F) \leq CM(\log M)^{2-Q},$$

if $F := \{x \in \Gamma; d(x, E) > M\}$, whereas $\min(\mathrm{diam}(E), \mathrm{diam}(F)) = M + 1$.

In the opposite direction we obtain a similar bound.

Proposition 5.4 *Under (LV_Q), (D) and (P_{Q-1}), one has*

$$\mathrm{Cap}_{Q-1}(E, F; B(x, Cr)) \geq c \min(\mathrm{diam}(E)(\log \mathrm{diam}(E) + 1)^{1-Q}, \mathrm{diam}(F)(\log \mathrm{diam}(F) + 1)^{1-Q}),$$

for some $C \geq 1$, $c > 0$, all $x \in X$, $r > 0$, E, F disjoint connected subsets of $B(x, r)$.

We leave it to the reader to modify the proof of Proposition 5.3 so as to obtain this estimate. Hint: replace (5.8) with

$$\sum_0^k i \leq k^2, \ \forall k \in \mathbb{N}^*.$$

We do not know if the exponent $1 - Q$ in Proposition 5.4 is the best one can obtain in a general setting. In $\mathbb{Z}^Q$, $Q \geq 3$, one can replace the exponent $1 - Q$ by the better exponent $2 - Q$ which is sharp by the above discussion. To see this we reason as follows. Given the sets E and F we pick balls B_1 and B_2 of radii $\mathrm{diam}(E)$ and $\mathrm{diam}(F)$, respectively, so that $E \subset B_1$ and $F \subset B_2$. Let f be a test function for the capacity. We may assume that $f = 1$ on E and $f = 0$ on F. Suppose first that $f_{2B_1} \geq 2/3$ and $f_{2B_2} \leq 1/3$. We compare these two averages by iterating the Poincaré inequality as in the second proof of Proposition 3.3 and arrive at

$$\sum_{B(x,Cr)} |\nabla f|^{Q-1} \geq c \min(\mathrm{diam}(E), \mathrm{diam}(F)).$$

The claim follows in this case. Suppose then that $f_{2B_1} \geq 2/3$; the other remaining case is handled the same way. Write $u(x) = (\mathrm{diam}(E) - d(x,B))_+/\mathrm{diam}(E)$ and $g(x) = u(x)(f(x) - f_{2B_1})$. Using (P_{Q-1}) one computes that

$$\||\nabla g|\|_{Q-1} \leq C\||\nabla f|\|_{Q-1}.$$

By the geometry of $\mathbb{Z}^Q$ one finds $\mathrm{diam}(E)/M$ disjoint copies of $\mathbb{Z}^{Q-1}$ in $\mathbb{Z}^Q$ that intersect E, where M depends only on Q. The claim follows by applying the pointwise estimate at the end of Section 4 to g on each of these copies and by summing over the disjoint copies.

Notice that the above argument used strongly the product structure of $\mathbb{Z}^Q$.

5.3 $1 \leq p < Q-1$

Here we assume $Q > 2$. The same Γ and E as for $p = Q-1$ with $F = \Gamma \setminus E$ and $f = \chi_E$ shows that the bound

$$\mathrm{Cap}_p(E, F; B(x, Cr)) \geq c \min(\mathrm{diam}(E), \mathrm{diam}(F))^{Q-p}$$

cannot hold. Indeed, in this example we only have linear growth. This growth turns out to hold in general.

Proposition 5.5 *Let $p \in [1, Q-1[$. Under (LV_Q), (D) and (P_p), one has*

$$\mathrm{Cap}_p(E, F; B(x, Cr)) \geq c \min(\mathrm{diam}(E), \mathrm{diam}(F)),$$

for some $C \geq 1$, $c > 0$, and all $x \in \Gamma$, $r > 0$, E, F disjoint connected subsets of $B(x,r)$.

Again, this is better than what the corresponding global Sobolev inequality would yield in terms of diameters. However, together with (UV_Q), the inequality

$$\mathrm{Cap}_p(E, F) \geq c \min(\mathrm{diam}(E), \mathrm{diam}(F)),$$

for connected sets, only gives back

$$\mathrm{Cap}_p(E, F) \geq c' \min(|E|, |F|)^{\frac{1}{Q}},$$

which is weaker than the information given by the global Sobolev inequality since $1 < Q - p$.

Proof: The proof of Proposition 5.3 is modified as follows. In the case where $|f_{2^k}(y)| \geq 1/3$ for every $y \in E$, we still take $\varepsilon = (p+1-Q)/p$, but now $\varepsilon < 0$, and $\sum_1^k 2^{\varepsilon i} \leq C(\varepsilon)$. One gets, for some i, $1 \leq i \leq k$,

$$C\||\nabla f\|_{p, B(y, 2^i)} \geq 2^{i/p},$$

and

$$C\||\nabla f|\|_{p,E}^p \geq \sum_i \mathrm{diam}(B_i) \geq \mathrm{diam}(E).$$

In the other cases, the same estimates as before hold, and they are stronger than the claim. □

5.4 Another L^p metric: the Grötsch invariant

The following invariant is considered in [33].

Definition 5.6 *For* $x, y \in \Gamma$,

$$g_p(x,y) = \inf\{\mathrm{Cap}_p(K);\ K \textit{ connected subset of } \Gamma \textit{ containing } x \textit{ and } y\}.$$

Here

$$\mathrm{Cap}_p(K) = \inf\{\||\nabla f|\|_p^p; f \in c_0(\Gamma), f\ |_K = 1\}.$$

It is not difficult to deduce an upper estimate for $g_p(x,y)$ from the upper estimate on the volume growth. As in Proposition 3.2, this is just a matter of choosing suitable test functions. We leave the proof to the reader.

Proposition 5.7 *Under* (UV_Q), $g_p(x,y) = 0$ *if* $p \geq Q$, *and for* $p \in [1, Q[$, *one has*

$$g_p(x,y) \leq Cd(x,y)^{Q-p},\ \forall\, x, y \in \Gamma.$$

Is is easy to see that if $p < Q$ and (S_Q^p) holds, then

$$g_p(x,y) \geq cd(x,y)^{\frac{Q-p}{p}}.$$

This can be improved in presence of Poincaré inequalities. Indeed, putting together the estimates in Propositions 5.3, 5.4 and 5.5, and using the assumption that Γ is connected and unbounded, we arrive at the following lower bounds on $g_p(x,y)$ for $1 < p < Q$.

Proposition 5.8 *Under* (LV_Q), (D) *and* (P_p), *one has*

$$g_p(x,y) \geq Cd(x,y)^{\min\{Q-p,1\}},\ \forall\, x, y \in \Gamma$$

when $1 \leq p < Q$, $p \neq Q-1$, *and*

$$g_{Q-1}(x,y) \geq Cd(x,y)(\log d(x,y))^{1-Q},\ \forall\, x, y \in \Gamma.$$

6 Glueing Poincaré inequalities

In this section we want to see (P_p) as a connectivity property, which is less and less stringent as p increases. As a consequence, it is more and more stable under glueing as p increases. Consider two graphs Γ_1, Γ_2 and let $A = \Gamma_1 \cap \Gamma_2$ be the set of vertices they have in common. The graph $\Gamma_1 \cup_A \Gamma_2$ obtained by glueing Γ_1 and Γ_2 along A is the graph whose vertices and edges are those of Γ_1 together with those of Γ_2. Assuming that Γ_1, Γ_2 satisfy (V_Q) and (P_p), one may wonder how big should A be so that $\Gamma_1 \cup_A \Gamma_2$ also satisfies (P_p)? The following theorem was obtained in [25].

Theorem 6.1 *Let* Γ_1, Γ_2 *satisfy* (V_Q) *and* (P_p), *for some* $p \in [1, +\infty[$. *Suppose* $A = \Gamma_1 \cap \Gamma_2$ *satisfies the following: there are numbers* $Q - p < \lambda \leq Q$ *and* $C \geq 1$ *so that*

$$H^{\infty}_{\lambda}(A \cap B(x,r)) \geq C^{-1} r^{\lambda}$$

for all balls $B(x,r)$ *that are centered at* A. *Then* $\Gamma_1 \cup_A \Gamma_2$ *satisfies* (P_p) *as well.*

The proof in [25] was given in the continuous setting but the result easily generalizes to the discrete setting. To see this, we will use Lemma 2.5. Indeed, let $x_1, x_2 \in \Gamma_1 \cup \Gamma_2$. Because both Γ_1 and Γ_2 satisfy (V_Q), so does also $\Gamma_1 \cup \Gamma_2$. By Lemma 2.5 and (P_p) for Γ_1 and Γ_2, we see that it suffices to show that

$$|f(x_1) - f(x_2)| \leq C\, d(x_1, x_2) \left((M_{Cd(x_1,x_2)} |\nabla f|^p)(x_1) + (M_{Cd(x_1,x_2)} |\nabla f|^p)(x_2) \right)^{1/p}$$

when $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$. We may assume that $f(x_1) = 0$ and that $f(x_2) = 1$. To this end, write $r = d(x_1, x_2)$, and pick $z \in A$ so that $d(x_1, z) + d(x_2, z) \leq 2r$. Once more, as in the second proof of Proposition 3.3, we have that

$$|f(x_i) - f_{B(x_i,r)}| \leq Cr \left(M_{Cr} |\nabla f|^p)(x_i) \right)^{1/p},$$

where $i = 1, 2$, and the maximal functions and averages are with respect to Γ_1, Γ_2. By (V_Q) we may replace the maximal functions with the maximal function with respect to $\Gamma_1 \cup \Gamma_2$. Thus the claim follows if either $f_{B(x_1,r)} \geq 1/5$ or $f_{B(x_2,r)} \leq 4/5$. Thus assume that this is not the case. Consider the averages $f_{B_{\Gamma_i}(z,2r)}$, $i = 1, 2$. Again, using (P_p) in Γ_1, Γ_2 and our assumption on the above two averages, we see that the claim follows if either $f_{B_{\Gamma_1}(z,2r)} \geq 2/5$ or $f_{B_{\Gamma_2}(z,2r)} \leq 3/5$. Thus suppose that both these inequalities fail. Let

$$A_1 = \{ w \in A \cap B(z, 2r);\ f(w) \geq 1/2 \}$$

and $A_2 = (A \setminus A_1) \cap B(z, 2r)$. Now,

$$H^{\lambda}_{\infty}(A \cap B(z, 2r)) \leq H^{\lambda}_{\infty}(A_1) + H^{\lambda}_{\infty}(A_2).$$

Suppose for instance that

$$H^{\lambda}_{\infty}(A_1) \geq H^{\lambda}_{\infty}(A \cap B(z, 2r))/2.$$

Now (V_Q), (P_p) on Γ_1 and our assumption on the size of A allow us to deduce from Lemma 4.4 that

$$\sum_{B(x_1, C'r)} |\nabla f|^p \geq \sum_{B_{\Gamma_1}(z, Cr)} |\nabla f|^p \geq c r^{Q-p} \geq c' V(x_1, C'r) d(x_1, x_2)^{-p}.$$

The claim follows. □

A much stronger result than indicated holds when $p > Q$. This can be proven by appropriately modifying the above argument and using results from Section 3.

Proposition 6.2 *Suppose* Γ_1, Γ_2 *satisfy* (V_Q) *and* (P_p). *If* $A = \Gamma_1 \cap \Gamma_2$ *is any nonempty set, then* $\Gamma_1 \cup_A \Gamma_2$ *satisfies* (P_p) *as well.*

The above Theorem is rather sharp. Indeed, Proposition 2.6 implies the necessity of the condition

$$H^{Q-p}_\infty(A \cap B(x,r)) \geq C^{-1} r^{Q-p}$$

for all balls $B(x,r)$ either in Γ_1 or in Γ_2 that are centered at A.

It would be convenient to have a sufficient condition in terms of the cardinality of $A \cap B(x,r)$ instead of the somewhat cumbersome Hausdorff content. One could expect that a uniform lower bound of the type $|A \cap B(x,r)| \geq Cr^\lambda$ for some $\lambda > Q-p$ would be sufficient. This is anyhow not enough as the following example shows. Let $A = \cup_j B((2^j, 0, \cdots, 0), 2^{j/n}) \subset \mathbb{Z}^n$, $n \geq 2$. Then it is easy to check that $|A \cap B(x,r)| \geq Cr$ for each $x \in A$ and all $r > 0$. However, the measure is badly distributed and we have

$$H^\lambda_\infty(A \cap B(0,2^j)) \leq C \sum_1^j 2^{j\lambda/n} \leq 2C 2^{j\lambda/n}$$

for each $\lambda > 0$. Thus (P_p) holds for $\mathbb{Z}^n \cup_A \mathbb{Z}^n$ for no $p < Q$. One can check that (P_Q) fails as well. A sufficient condition can be given in terms of two sided estimates on $|A \cap B(x,r)|$.

Proposition 6.3 *Let* Γ_1, Γ_2 *satisfy* (V_Q) *and* (P_p). *Suppose* $A = \Gamma_1 \cap \Gamma_2$ *satisfies the following: there are numbers* $\lambda > Q-p$ *and* $C \geq 1$ *so that*

$$C^{-1} r^\lambda \leq |A \cap B(x,r)| \leq Cr^\lambda$$

for all balls $B(x,r)$ *that are centered at* A. *Then* $\Gamma_1 \cup_A \Gamma_2$ *satisfies* (P_p) *as well.*

To see that this condition is sufficient, we argue as follows. By Theorem 6.1, we only need to show that

$$H^\lambda_\infty(A \cap B(x,r)) \geq C^{-1} r^\lambda.$$

To this end, let $B(x_i, r_i)$ cover $A \cap B(x,r)$. Then

$$\sum r_i^\lambda \geq C^{-1} |A \cap B(x_i, r)| \geq C^{-1} |A \cap B(x,r)| \geq C^{-2} r^\lambda,$$

as desired. □

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